Western Mathematical Olympiad 2001
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by orl

Day 1

1 The sequence $\{x_n\}$ satisfies $x_1 = \frac{1}{2}, x_{n+1} = x_n + \frac{x_n^2}{n^2}$. Prove that $x_{2001} < 1001$.

2 $ABCD$ is a rectangle of area 2. P is a point on side CD and Q is the point where the incircle of $\triangle PAB$ touches the side AB. The product $PA \cdot PB$ varies as $ABCD$ and P vary. When $PA \cdot PB$ attains its minimum value,

a) Prove that $AB \geq 2BC$,
b) Find the value of $AQ \cdot BQ$.

3 Let n, m be positive integers of different parity, and $n > m$. Find all integers x such that $\frac{x^{2^n}-1}{x^{2^m}-1}$ is a perfect square.

4 Let x, y, z be real numbers such that $x + y + z \geq xyz$. Find the smallest possible value of $\frac{x^2+y^2+z^2}{xyz}$.

Day 2

1 Find all real numbers x such that $\lfloor x^3 \rfloor = 4x + 3$.

2 P is a point on the exterior of a circle centered at O. The tangents to the circle from P touch the circle at A and B. Let Q be the point of intersection of PO and AB. Let CD be any chord of the circle passing through Q. Prove that $\triangle PAB$ and $\triangle PCD$ have the same incentre.

3 Find, with proof, all real numbers $x \in [0, \frac{\pi}{2}]$, such that $(2 - \sin 2x)\sin(x + \frac{\pi}{4}) = 1$.

4 We call $A_1, A_2, \ldots, A_n$ an n-division of A if

(i) $A_1 \cap A_2 \cap \cdots \cap A_n = A$,
(ii) $A_i \cap A_j \neq \emptyset$.

Find the smallest positive integer m such that for any 14-division $A_1, A_2, \ldots, A_{14}$ of $A = \{1, 2, \ldots, m\}$, there exists a set A_i $(1 \leq i \leq 14)$ such that there are two elements a, b of A_i such that $b < a \leq \frac{4}{3}b$.