Western Mathematical Olympiad 2002

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by Fang-jh, April

Day 1

1 Find all positive integers n such that $n^4 - 4n^3 + 22n^2 - 36n + 18$ is a perfect square.

2 Let O be the circumcenter of acute triangle ABC. Point P is in the interior of triangle AOB. Let D, E, F be the projections of P on the sides BC, CA, AB, respectively. Prove that the parallelogram consisting of FE and FD as its adjacent sides lies inside triangle ABC.

3 In the complex plane, consider squares having the following property: the complex numbers its vertex correspond to are exactly the roots of integer coefficients equation $x^4 + px^3 + qx^2 + rx + s = 0$. Find the minimum of square areas.

4 Let n be a positive integer, let the sets $A_1, A_2, \cdots, A_{n+1}$ be non-empty subsets of the set $\{1, 2, \cdots, n\}$. prove that there exist two disjoint non-empty subsets of the set $\{1, 2, \cdots, n+1\}$: $\{i_1, i_2, \cdots, i_k\}$ and $\{j_1, j_2, \cdots, j_m\}$ such that $A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k} = A_{j_1} \cup A_{j_2} \cup \cdots \cup A_{j_m}$.

Day 2

1 Given a trapezoid $ABCD$ with $AD \parallel BC, E$ is a moving point on the side AB, let O_1, O_2 be the circumcenters of triangles AED, BEC, respectively. Prove that the length of O_1O_2 is a constant value.

2 Given a positive integer n, find all integers $(a_1, a_2, \cdots, a_n)$ satisfying the following conditions: $(1): a_1 + a_2 + \cdots + a_n \geq n^2; (2): a_1^2 + a_2^2 + \cdots + a_n^2 \leq n^3 + 1$.

3 Assume that α and β are two roots of the equation: $x^2 - x - 1 = 0$. Let $a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, n = 1, 2, \cdots$.
(1) Prove that for any positive integer n, we have $a_{n+2} = a_{n+1} + a_n$.
(2) Find all positive integers a and b, $a < b$, satisfying $b \mid a_n - 2na^n$ for any positive integer n.

4 Assume that $S = (a_1, a_2, \cdots, a_n)$ consists of 0 and 1 and is the longest sequence of number, which satisfies the following condition: Every two sections of successive 5 terms in the sequence of numbers S are different, i.e., for arbitrary $1 \leq i < j \leq n-4$, $(a_i, a_{i+1}, a_{i+2}, a_{i+3}, a_{i+4})$ and $(a_j, a_{j+1}, a_{j+2}, a_{j+3}, a_{j+4})$ are different. Prove that the first four terms and the last four terms in the sequence are the same.