Saint Petersburg Mathematical Olympiad 2010

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– **Grade 11**

1 Solve in positives
$$x^y = z, y^z = x, z^x = y$$

2 ABC is triangle with $AB = BC$. X, Y are midpoints of AC and AB. Z is base of perpendicular from B to CY. Prove, that circumcenter of XYZ lies on AC

3 There are 2009 cities in country, and every two are connected by road. Businessman and Road Ministry play next game. Every morning Businessman buys one road and every evening Ministry destroys 10 free roads. Can Business create cyclic route without self-intersections through exactly 75 different cities?

4 Natural number N is given. Let p_N - biggest prime, that $\leq N$. On every move we replace N by $N - p_N$. We repeat this until we get 0 or 1. If we get 1 then N is called as good, else is bad. For example, 95 is good because we get $95 \to 6 \to 1$.
Prove that among numbers from 1 to 1000000 there are between one quarter and half good numbers

5 $SABCD$ is quadrangular pyramid. Lateral faces are acute triangles with orthocenters lying in one plane. $ABCD$ is base of pyramid and AC and BD intersects at P, where SP is height of pyramid. Prove that $AC \perp BD$

6 For positive numbers is true that
$$ab + ac + bc = a + b + c$$
Prove
$$a + b + c + 1 \geq 4abc$$

7 600 integer numbers from $[1, 1000]$ colored in red. Natural segment $[n, k]$ is called yummy if for every natural t from $[1, k - n]$ there are two red numbers a, b from $[n, k]$ and $b - a = t$.
Prove that there is yummy segment with $[a, b]$ with $b - a \geq 199$

– **Grade 10**

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1 $f(x)$ is square trinomial. Is it always possible to find polynomial $g(x)$ with fourth degree, such that $f(g(x))=0$ has not roots?

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2 There are 10 consecutive 30-digit numbers. We write the biggest divisor for every number (divisor is not equal number). Prove that some written numbers ends with same digit.

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3 M,N are midpoints of AB and CD for convex quadrilateral $ABCD$. Points X and Y are on AD and BC and $XD=3AX, YC=3BY$. $\angle MXA=\angle MYB=90$.
Prove that $\angle XMN=\angle ABC$

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4 There are 2010 cities in country, and 3 roads go from every city. President and Prime Minister play next game.
They sell roads by turn to one of 3 companies(one road is one turn). President will win, if three roads from some city are sold to different companies.
Who will win?

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5 Same as Grade 11 P2

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6 Natural number N is given. Let p_N - biggest prime, that $\leq N$. On every move we replace N by $N-p_N$. We repeat this until we get 0 or 1. Prove that exists such number N, that we need exactly 1000 turns to make 0

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7 200×200 square is colored in chess order. In one move we can take every 2×3 rectangle and change color of all its cells. Can we make all cells of square in same color ?

– **Grade 9**

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1 Chess king is standing in some square of chessboard. Every sunday it is moved to one square by diagonal, and every another day it is moved to one square by horisontal or vertical. What maximal numbers of moves can be made ?

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2 Same as Grade10 P3

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3 a is irrational , but a and a^3-6a are roots of square polynomial with integer coefficients.Find a

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4 A -is 20-digit number. We write 101 numbers A then erase last 11 digits. Prove that this 2009-digit number can not be degree of 2

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5 There are 2010 cities in country, and every two are connected by road. Businessman and Road Ministry play next game. Every morning Businessman buys one road and every evening Ministry destroys 10 free roads. Can Business create cyclic route without self-intersections through exactly 11 different cities?

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6 For positive is true

$$\frac{3}{abc} \geq a+b+c$$

Prove

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq a+b+c$$

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7 Incircle of ABC tangent AB, AC, BC in C_1, B_1, A_1. AA_1 intersect incircle in E. N is midpoint B_1A_1. M is symmetric to N relatively AA_1. Prove that $\angle EMC = 90$

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