Saint Petersburg Mathematical Olympiad 2009
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by RagvaloD

–	**Grade 11**
1	$f(x)=ax^2+bx+c; a,b,c$ are reals. $M=\{f(2n) \mid n$ is integer$\}, N=\{f(2n+1) \mid n$ is integer$\}$ Prove that $M=N$ or $M\cap N=\emptyset$
2	$[x,y]-[x,z]=y-z$ and $x\neq y\neq z\neq x$ Prove, that $x\mid y, x\mid z$
3	Streets of Moscow are some circles (rings) with common center O and some straight lines from center O to external ring. Point A,B - two crossroads on external ring. Three friends want to move from A to B. Dima goes by external ring, Kostya goes from A to O then to B. Sergey says, that there is another way, that is shortest. Prove, that he is wrong.
4	From 2008×2008 square we remove one corner cell 1×1. Is number of ways to divide this figure to corners from 3 cells odd or even ?
5	O -circumcenter of $ABCD$. AC and BD intersect in E, AD and BC in F. X,Y - midpoints of AD and BC. O_1 -circumcenter of EXY. Prove that $OF \parallel O_1E$
6	Some cities in country are connected by road, and from every city goes ≥ 2008 roads. Every road is colored in one of two colors. Prove, that exists cycle without self-intersections ,where ≥ 504 roads and all roads are same color.
7	$f(x)=x^2+x$ $b_1,...,b_{10000}>0$ and $\lvert b_{n+1}-f(b_n)\rvert\leq\frac{1}{1000}$ for $n=1,...,9999$ Prove, that there is such $a_1>0$ that $a_{n+1}=f(a_n); n=1,...,9999$ and $\lvert a_n-b_n\rvert<\frac{1}{100}$
–	**Grade 10**
1	x,y are naturals. $GCM(x^7,y^4)*GCM(x^8,y^5)=xy$ Prove that xy is cube
2	$ABCD$ is convex quadrilateral with $AB=CD$. AC and BD intersect in O. X,Y,Z,T are midpoints of BC,AD,AC,BD. Prove, that circumcenter of OZT lies on XY.
3	$f(x),g(x),h(x)$ are square trinomials with discriminant, that equals 2. And $f(x)+g(x),f(x)+h(x),g(x)+h(x)$ are square trinomials with discriminant, that equals 1. Prove,that $f(x)+g(x)+h(x)$ has not roots.

4 Same as Grade 11 P3

5 Call a set of some cells in infinite chess field as board. Set of rooks on the board call as awesome if no one rook can beat another, but every empty cell is under rook attack. There are awesome set with 2008 rooks and with 2010 rooks. Prove, that there are awesome set with 2009 rooks.

6 (x_n) is sequence, such that $x_{n+2}=|x_{n+1}|-x_n$. Prove, that it is periodic.

7 Points Y,X lies on AB,BC of $\triangle ABC$ and X,Y,A,C are concyclic. AX and CY intersect in O. Points M,N are midpoints of AC and XY. Prove, that BO is tangent to circumcircle of $\triangle MON$

– **Grade 9**

1 b,c are naturals. $b|c+1$ Prove that exists such natural x,y,z that $x+y=bz, xy=cz$

2 There are 40 members of jury, that want to choose problem for contest. There are list with 30 problems. They want to find such problem, that can be solved at least half members , but not all. Every member solved 26 problems, and every two members solved different sets of problems.
Prove, that they can find problem for contest.

3 Same as Grade 11,P2

4 Points A_1 and C_1 are on BC and AB of acute-angled triangle ABC . AA_1 and CC_1 intersect in K. Circumcircles of AA_1B, CC_1B intersect in P - incenter of AKC.
Prove, that P - orthocenter of ABC

5 ABC is acute-angled triangle. AA_1,BB_1,CC_1 are altitudes. X,Y - midpoints of AC_1,A_1C. $XY=BB_1$.
Prove that one side of ABC in $\sqrt{2}$ greater than other side.

6 Same as Grade 10,P5

7 Discriminants of square trinomials $f(x),g(x),h(x),f(x)+g(x),f(x)+h(x),g(x)+h(x)$ equals 1.
Prove that $f(x)+h(x)+g(x)\equiv 0$