Western Mathematical Olympiad 2003

www.artofproblemsolving.com/community/c5197

by orl

Day 1

1 Place the numbers $1, 2, 3, 4, 5, 6, 7, 8$ at the vertices of a cuboid such that the sum of any 3 numbers on a side is not less than 10. Find the smallest possible sum of the 4 numbers on a side.

2 Let $a_1, a_2, \ldots, a_{2n}$ be $2n$ real numbers satisfying the condition $\sum_{i=1}^{2n-1}(a_{i+1}-a_i)^2 = 1$. Find the greatest possible value of $(a_{n+1}+a_{n+2}+\ldots+a_{2n})-(a_1+a_2+\ldots+a_n)$.

3 Let n be a given positive integer. Find the smallest positive integer u_n such that for any positive integer d, in any u_n consecutive odd positive integers, the number of them that can be divided by d is not smaller than the number of odd integers among $1, 3, 5, \ldots, 2n-1$ that can be divided by d.

4 Given that the sum of the distances from point P in the interior of a convex quadrilateral $ABCD$ to the sides AB, BC, CD, DA is a constant, prove that $ABCD$ is a parallelogram.

Day 2

1 The sequence $\{a_n\}$ satisfies $a_0 = 0, a_{n+1} = ka_n + \sqrt{(k^2-1)a_n^2+1}, n = 0, 1, 2, \ldots$, where k is a fixed positive integer. Prove that all the terms of the sequence are integral and that $2k$ divides $a_{2n}, n = 0, 1, 2, \ldots$.

2 A circle can be inscribed in the convex quadrilateral $ABCD$. The incircle touches the sides AB, BC, CD, DA at A_1, B_1, C_1, D_1 respectively. The points E, F, G, H are the midpoints of $A_1B_1, B_1C_1, C_1D_1, D_1A_1$ respectively. Prove that the quadrilateral $EFGH$ is a rectangle if and only if A, B, C, D are concyclic.

3 The non-negative numbers $x_1, x_2, \ldots, x_5$ satisfy $\sum_{i=1}^{5}\frac{1}{1+x_i} = 1$. Prove that $\sum_{i=1}^{5}\frac{x_i}{4+x_i^2} \leq 1$.

4 1650 students are arranged in 22 rows and 75 columns. It is known that in any two columns, the number of pairs of students in the same row and of the same sex is not greater than 11. Prove that the number of boys is not greater than 928.