Western Mathematical Olympiad 2004

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Day 1

1 Find all integers n, such that the following number is a perfect square

$$N = n^4 + 6n^3 + 11n^2 + 3n + 31.$$

2 Let $ABCD$ be a convex quadrilateral, I_1 and I_2 be the incenters of triangles ABC and DBC respectively. The line I_1I_2 intersects the lines AB and DC at points E and F respectively. Given that AB and CD intersect in P, and $PE = PF$, prove that the points A, B, C, D lie on a circle.

3 Find all reals k such that

$$a^3 + b^3 + c^3 + d^3 + 1 \geq k(a + b + c + d)$$

holds for all $a, b, c, d \geq -1$.

Edited by orl.

4 Let $\mathbb{N}$ be the set of positive integers. Let $n \in \mathbb{N}$ and let $d(n)$ be the number of divisors of n. Let $\varphi(n)$ be the Euler-totient function (the number of co-prime positive integers with n, smaller than n).

Find all non-negative integers c such that there exists $n \in \mathbb{N}$ such that

$$d(n) + \varphi(n) = n + c,$$

and for such c find all values of n satisfying the above relationship.

Day 2

1 The sequence $\{a_n\}_n$ satisfies the relations $a_1 = a_2 = 1$ and for all positive integers n,

$$a_{n+2} = \frac{1}{a_{n+1}} + a_n.$$

Find a_{2004}.

2 All the grids of a $m \times n$ chess board ($m, n \geq 3$), are colored either with red or with blue. Two adjacent grids (having a common side) are called a "good couple" if they have different colors. Suppose there are S "good couples". Explain how to determine whether S is odd or even. Is it prescribed by
some specific color grids? Justify your answers.

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3 Let ℓ be the perimeter of an acute-angled triangle ABC which is not an equilateral triangle. Let P be a variable points inside the triangle ABC, and let D, E, F be the projections of P on the sides BC, CA, AB respectively. Prove
that

$$2(AF + BD + CE) = \ell$$

if and only if P is collinear with the incenter and the circumcenter of the triangle ABC.

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4 Suppose that a, b, c are positive real numbers, prove that

$$1 < \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{b^2+c^2}} + \frac{c}{\sqrt{c^2+a^2}} \leq \frac{3\sqrt{2}}{2}$$
