Western Mathematical Olympiad 2005

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Day 1 November 5th

1 It is known that $a^{2005} + b^{2005}$ can be expressed as the polynomial of $a + b$ and ab. Find the coefficients' sum of this polynomial.

2 Given three points P, A, B and a circle such that the lines PA and PB are tangent to the circle at the points A and B, respectively. A line through the point P intersects that circle at two points C and D. Through the point B, draw a line parallel to PA; let this line intersect the lines AC and AD at the points E and F, respectively. Prove that $BE = BF$.

3 Set $S = \{1, 2, 3, ..., 2005\}$. If among any n pairwise coprime numbers in S there exists at least a prime number, find the minimum of n.

4 Given is the positive integer $n > 2$. Real numbers $| x_i | \leq 1$ $(i = 1, 2, ..., n)$ satisfying $| \sum_{i=1}^{n} x_i | > 1$. Prove that there exists positive integer k such that $| \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i | \leq 1$.

Day 2 November 6th

5 Circles $C(O_1)$ and $C(O_2)$ intersect at points A, B. CD passing through point O_1 intersects $C(O_1)$ at point D and tangents $C(O_2)$ at point C. AC tangents $C(O_1)$ at A. Draw $AE \perp CD$, and AE intersects $C(O_1)$ at E. Draw $AF \perp DE$, and AF intersects DE at F. Prove that BD bisects AF.

6 In isosceles right-angled triangle ABC, $CA = CB = 1$. P is an arbitrary point on the sides of ABC. Find the maximum of $PA \cdot PB \cdot PC$.

7 If a, b, c are positive reals such that $a + b + c = 1$, prove that

$$10(a^3 + b^3 + c^3) - 9(a^5 + b^5 + c^5) \geq 1.$$

8 For n people, if it is known that

(a) there exist two people knowing each other among any three people, and

(b) there exist two people not knowing each other among any four people.

Find the maximum of n.

Here, we assume that if A knows B, then B knows A.
