Western Mathematical Olympiad 2006

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by Jumbler

Day 1

1 Let n be a positive integer with $n \geq 2$, and $0 < a_1, a_2, ..., a_n < 1$. Find the maximum value of the sum $\sum_{i=1}^{n}(a_i(1-a_{i+1}))^{\frac{1}{6}}$
where $a_{n+1} = a_1$

2 Find the smallest positive real k satisfying the following condition: for any given four DIFFERENT real numbers a, b, c, d, which are not less than k, there exists a permutation (p, q, r, s) of (a, b, c, d), such that the equation $(x^2 + px + q)(x^2 + rx + s) = 0$ has four different real roots.

3 In $\triangle PBC$, $\angle PBC = 60^o$, the tangent at point P to the circumcircle$g$ of $\triangle PBC$ intersects with line CB at A. Points D and E lie on the line segment PA and g respectively, satisfying $\angle DBE = 90^o$, $PD = PE$. BE and PC meet at F. It is known that lines AF, BP, CD are concurrent.
a) Prove that BF bisect $\angle PBC$
b) Find $\tan \angle PCB$

4 Assuming that the positive integer a is not a perfect square, prove that for any positive integer n, the sum $S_n = \sum_{i=1}^{n}\{a^{\frac{1}{2}}\}^i$ is irrational.

Day 2

1 Let $S = \{n | n-1, n, n+1$ can be expressed as the sum of the square of two positive integers.$\}$. Prove that if n in S, n^2 is also in S.

2 AB is a diameter of the circle O, the point C lies on the line AB produced. A line passing though C intersects with the circle O at the point D and E. OF is a diameter of circumcircle O_1 of $\triangle BOD$. Join CF and produce, cutting the circle O_1 at G. Prove that points O, A, E, G are concyclic.

3 Let k be a positive integer not less than 3 and x a real number. Prove that if $\cos(k-1)x$ and $\cos kx$ are rational, then there exists a positive integer $n > k$, such that both $\cos(n-1)x$ and $\cos nx$ are rational.

4 Given a positive integer $n \geq 2$, let B_1, B_2, ..., B_n denote n subsets of a set X such that each B_i contains exactly two elements. Find the minimum value of $|X|$ such that for any such choice of subsets B_1, B_2, ..., B_n, there exists a subset Y of X such that:

(1) $|Y|=n$;
(2) $|Y \cap B_i| \leq 1$ for every $i \in \{1, 2, ..., n\}$.