Western Mathematical Olympiad 2007

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Day 1

1 Let set $T = \{1,2,3,4,5,6,7,8\}$. Find the number of all nonempty subsets A of T such that $3|S(A)$ and $5 \nmid S(A)$, where $S(A)$ is the sum of all the elements in A.

2 Let C and D be two intersection points of circle O_1 and circle O_2. A line, passing through D, intersects the circle O_1 and the circle O_2 at the points A and B respectively. The points P and Q are on circles O_1 and O_2 respectively. The lines PD and AC intersect at H, and the lines QD and BC intersect at M. Suppose that O is the circumcenter of the triangle ABC. Prove that $OD \perp MH$ if and only if P, Q, M and H are concyclic.

3 Let a, b, c be real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{5a^2-4a+11}+\frac{1}{5b^2-4b+11}+\frac{1}{5c^2-4c+11}\leq\frac{1}{4}$$

4 Let O be an interior point of the triangle ABC. Prove that there exist positive integers p, q and r such that

$$|p\cdot\overrightarrow{OA}+q\cdot\overrightarrow{OB}+r\cdot\overrightarrow{OC}|<\frac{1}{2007}$$

Day 2

1 Is there a triangle with sides of integer lengths such that the length of the shortest side is 2007 and that the largest angle is twice the smallest?

2 Find all natural numbers n such that there exist $x_1, x_2, \dots, x_n, y \in \mathbb{Z}$ where $x_1, x_2, \dots, x_n, y \neq 0$ satisfying:

$$x_1 + x_2 + \ldots + x_n = 0$$
$$ny^2 = x_1^2 + x_2^2 + \ldots + x_n^2$$

3 Let P be an interior point of an acute angled triangle ABC. The lines AP, BP, CP meet BC, CA, AB at points D, E, F respectively. Given that triangle $\triangle DEF$ and $\triangle ABC$ are similar, prove that P is the centroid of $\triangle ABC$.

4 A circular disk is partitioned into $2n$ equal sectors by n straight lines through its center. Then, these $2n$ sectors are colored in such a way that exactly n of the sectors are colored in blue, and the other n sectors are colored in red. We number the red sectors with numbers from 1 to n in counter-clockwise direction (starting at some of these red sectors), and then we number the blue sectors with numbers from 1 to n in clockwise direction (starting at some of these blue sectors).

Prove that one can find a half-disk which contains sectors numbered with all the numbers from 1 to n (in some order). (In other words, prove that one can find n consecutive sectors which are numbered by all numbers 1, 2, ..., n in some order.)
n white and n black balls are placed at random on the circumference of a circle.Starting from a certain white ball,number all white balls in a clockwise direction by $1, 2, \dots, n$. Likewise number all black balls by $1, 2, \dots, n$ in anti-clockwise direction starting from a certain black ball.Prove that there exists a chain of n balls whose collection of numbering forms the set $\{1, 2, 3 \dots, n\}$.