Western Mathematical Olympiad 2008
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Day 1

1 A sequence of real numbers $\{a_n\}$ is defineds by $a_0 \neq 0, 1$, $a_1 = 1 - a_0$,$a_{n+1} = 1 - a_n(1 - a_n)$, $n = 1, 2, ...$

Prove that for any positive integer n, we have $a_0a_1...a_n(\frac{1}{a_0} + \frac{1}{a_1} + ... + \frac{1}{a_n}) = 1$

2 In triangle ABC, $AB = AC$, the inscribed circle I touches BC, CA, AB at points D, E and F respectively. P is a point on arc EF opposite D. Line BP intersects circle I at another point Q, lines EP, EQ meet line BC at M, N respectively. Prove that
(1) P, F, B, M concyclic
(2)$\frac{EM}{EN} = \frac{BD}{BP}$

(P.S. Can anyone help me with using GeoGebra, the incircle function of the plugin doesn't work with my computer.)

3 Given an integer $m \geq 2$, m positive integers $a_1, a_2, ...a_m$. Prove that there exist infinitely many positive integers n, such that $a_1 1^n + a_2 2^n + ... + a_m m^n$ is composite.

4 Given an integer $m \geq 2$, and two real numbers a, b with $a > 0$ and $b \neq 0$. The sequence $\{x_n\}$ is such that $x_1 = b$ and $x_{n+1} = ax_n^m + b$, $n = 1, 2, ...$ Prove that
(1)when $b < 0$ and m is even, the sequence is bounded if and only if $ab^{m-1} \geq -2$;
(2)when $b < 0$ and m is odd, or when $b > 0$ the sequence is bounded if and only if $ab^{m-1} \geq \frac{(m-1)^{m-1}}{m^m}$.

Day 2

1 Four frogs are positioned at four points on a straight line such that the distance between any two neighbouring points is 1 unit length. Suppose the every frog can jump to its corresponding point of reflection, by taking any one of the other 3 frogs as the reference point. Prove that, there is no such case that the distance between any two neighbouring points, where the frogs stay, are all equal to 2008 unit length.

2 Given $x, y, z \in (0, 1)$ satisfying that $\sqrt{\frac{1-x}{yz}} + \sqrt{\frac{1-y}{xz}} + \sqrt{\frac{1-z}{xy}} = 2$.
Find the maximum value of xyz.

3 For a given positive integer n, find the greatest positive integer k, such that there exist three sets of k non-negative distinct integers, $A = \{x_1, x_2, \cdots, x_k\}, B = \{y_1, y_2, \cdots, y_k\}$ and $C = \{z_1, z_2, \cdots, z_k\}$ with $x_j + y_j + z_j = n$ for any $1 \leq j \leq k$.

[Moderator edit: LaTeXified]

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4 Let P be an interior point of a regular n-gon $A_1A_2...A_n$, the lines A_iP meet the regular n-gon at another point B_i, where $i = 1, 2, ..., n$. Prove that sums of all $PA_i \geq$ sum of all PB_i.
