Western Mathematical Olympiad 2009

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by horizon

Day 1

1 Let M be the set of the real numbers except for finitely many elements. Prove that for every positive integer n there exists a polynomial $f(x)$ with $\deg f = n$, such that all the coefficients and the n real roots of f are all in M.

2 Given an integer $n \geq\ 3$, find the least positive integer k, such that there exists a set A with k elements, and n distinct reals $x_{1},x_{2},\ldots,x_{n}$ such that $x_{1}+x_{2},x_{2}+x_{3},\ldots,x_{n-1}+x_{n},x_{n}+x_{1}$ all belong to A.

3 Let H be the orthocenter of acute triangle ABC and D the midpoint of BC. A line through H intersects AB,AC at F,E respectively, such that $AE = AF$. The ray DH intersects the circumcircle of $\triangle ABC$ at P. Prove that P,A,E,F are concyclic.

4 Prove that for every given positive integer k, there exist infinitely many n, such that $2^{n}+3^{n}-1,2^{n}+3^{n}-2,\ldots,2^{n}+3^{n}-k$ are all composite numbers.

Day 2

1 Define a sequence $(x_{n})_{n\geq 1}$ by taking $x_{1}\in\left\{5,7\right\}$; when $k\ge 1$, $x_{k+1}\in\left\{5^{x_{k}},7^{x_{k}}\right\}$. Determine all possible last two digits of x_{2009}.

2 Given an acute triangle ABC, D is a point on BC. A circle with diameter BD intersects line AB,AD at X,P respectively (different from B,D).The circle with diameter CD intersects AC,AD at Y,Q respectively (different from C,D). Draw two lines through A perpendicular to PX,QY, the feet are M,N respectively.Prove that $\triangle AMN$ is similar to $\triangle ABC$ if and only if AD passes through the circumcenter of $\triangle ABC$.

3 A total of n people compete in a mathematical match which contains 15 problems where $n>12$. For each problem, 1 point is given for a right answer and 0 is given for a wrong answer. Analysing each possible situation, we find that if the sum of points every group of 12 people get is no less than 36, then there are at least 3 people that got the right answer of a certain problem, among the n people. Find the least possible n.

4 The real numbers $a_{1},a_{2},\ldots,a_{n}$ where $n\ge 3$ are such that $\sum_{i=1}^{n}a_{i}=0$ and $2a_{k}\le\ a_{k-1}+a_{k+1}$ for all $k=2,3,\ldots,n-1$. Find the least $f(n)$ such that, for all $k\in\left\{1,2,\ldots,n\right\}$, we have $|a_{k}|\le f(n)\max\left\{|a_{1}|,|a_{n}|\right\}$.

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