Western Mathematical Olympiad 2010

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1 Suppose that m and k are non-negative integers, and $p = 2^{2^m} + 1$ is a prime number. Prove that
(a) $2^{2^{m+1}p^k} \equiv 1 \pmod{p^{k+1}}$;
(b) $2^{m+1}p^k$ is the smallest positive integer n satisfying the congruence equation $2^n \equiv 1 \pmod{p^{k+1}}$.

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2 AB is a diameter of a circle with center O. Let C and D be two different points on the circle on the same side of AB, and the lines tangent to the circle at points C and D meet at E. Segments AD and BC meet at F. Lines EF and AB meet at M. Prove that E, C, M and D are concyclic.

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3 Determine all possible values of positive integer n, such that there are n different 3-element subsets $A_1, A_2, ..., A_n$ of the set $\{1, 2, ..., n\}$, with $|A_i \cap A_j| \neq 1$ for all $i \neq j$.

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4 Let $a_1, a_2, .., a_n, b_1, b_2, ..., b_n$ be non-negative numbers satisfying the following conditions simultaneously:

(1) $\sum_{i=1}^{n}(a_i + b_i) = 1$;

(2) $\sum_{i=1}^{n} i(a_i - b_i) = 0$;

(3) $\sum_{i=1}^{n} i^2(a_i + b_i) = 10$.

Prove that $\max\{a_k, b_k\} \leq \frac{10}{10 + k^2}$ for all $1 \leq k \leq n$.

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5 Let k be an integer and $k > 1$. Define a sequence $\{a_n\}$ as follows:

$a_0 = 0$,

$a_1 = 1$, and

$a_{n+1} = ka_n + a_{n-1}$ for $n = 1, 2, ...$

Determine, with proof, all possible k for which there exist non-negative integers $l, m (l \neq m)$ and positive integers p, q such that $a_l + ka_p = a_m + ka_q$.

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6 ΔABC is a right-angled triangle, $\angle C = 90^\circ$. Draw a circle centered at B with radius BC. Let D be a point on the side AC, and DE is tangent to the circle at E. The line through C perpendicular to AB meets line BE at F. Line AF meets DE at point G. The line through A parallel to BG meets DE at H. Prove that $GE = GH$.

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7 There are n $(n \geq 3)$ players in a table tennis tournament, in which any two players have a match. Player A is called not out-performed by player B, if at least one of player A's losers is not a B's loser.

Determine, with proof, all possible values of n, such that the following case could happen: after finishing all the matches, every player is not out-performed by any other player.

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8 Determine all possible values of integer k for which there exist positive integers a and b such that $\dfrac{b+1}{a} + \dfrac{a+1}{b} = k$.
