Western Mathematical Olympiad 2011

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Day 1

1 Given that $0 < x, y < 1$, find the maximum value of $\frac{xy(1-x-y)}{(x+y)(1-x)(1-y)}$

2 Let M be a subset of $\{1, 2, 3...2011\}$ satisfying the following condition:
For any three elements in M, there exist two of them a and b such that $a|b$ or $b|a$.
Determine the maximum value of $|M|$ where $|M|$ denotes the number of elements in M

3 Let $n \geq 2$ be a given integer $a)$ Prove that one can arrange all the subsets of the set $\{1, 2..., n\}$ as a sequence of subsets $A_1, A_2, \cdots, A_{2^n}$, such that $|A_{i+1}| = |A_i| + 1$ or $|A_i| - 1$ where $i = 1, 2, 3, \cdots, 2^n$ and $A_{2^n+1} = A_1$ $b)$ Determine all possible values of the sum $\sum_{i=1}^{2^n} (-1)^i S(A_i)$ where $S(A_i)$ denotes the sum of all elements in A_i and $S(\emptyset) = 0$, for any subset sequence $A_1, A_2, \cdots, A_{2^n}$ satisfying the condition in $a)$

4 In a circle Γ_1, centered at O, AB and CD are two unequal in length chords intersecting at E inside Γ_1. A circle Γ_2, centered at I is tangent to Γ_1 internally at F, and also tangent to AB at G and CD at H. A line l through O intersects AB and CD at P and Q respectively such that $EP = EQ$. The line EF intersects l at M. Prove that the line through M parallel to AB is tangent to Γ_1

Day 2

1 Does there exist any odd integer $n \geq 3$ and n distinct prime numbers $p_1, p_2, \cdots p_n$ such that all $p_i + p_{i+1} (i = 1, 2, \cdots, n$ and $p_{n+1} = p_1)$ are perfect squares?

2 Let $a, b, c > 0$, prove that

$$\frac{(a-b)^2}{(c+a)(c+b)} + \frac{(b-c)^2}{(a+b)(a+c)} + \frac{(c-a)^2}{(b+c)(b+a)} \geq \frac{(a-b)^2}{a^2+b^2+c^2}$$

3 In triangle ABC with $AB > AC$ and incenter I, the incircle touches BC, CA, AB at D, E, F respectively. M is the midpoint of BC, and the altitude at A meets BC at H. Ray AI meets lines DE and DF at K and L, respectively. Prove that the points M, L, H, K are concyclic.

4 Find all pairs of integers (a, b) such that $n|(a^n + b^{n+1})$ for all positive integer n