Western Mathematical Olympiad 2013

www.artofproblemsolving.com/community/c5207

by sqing, 61plus, monsterrr

1 Does there exist any integer a,b,c such that $a^2bc+2, ab^2c+2, abc^2+2$ are perfect squares?

---

2 Let the integer $n\ge 2$, and the real numbers $x_1,x_2,\cdots,x_n\in[0,1]$.Prove that

$$\sum_{1\le k<j\le n} kx_kx_j \le \frac{n-1}{3}\sum_{k=1}^{n} kx_k.$$

---

3 Let ABC be a triangle, and B_1,C_1 be its excenters opposite B,C. B_2,C_2 are reflections of B_1,C_1 across midpoints of AC,AB. Let D be the extouch at BC. Show that AD is perpendicular to B_2C_2

---

4 There are n coins in a row, $n\ge 2$. If one of the coins is head, select an odd number of consecutive coins (or even 1 coin) with the one in head on the leftmost, and then flip all the selected coins upside down simultaneously. This is a $move$. No move is allowed if all n coins are tails. Suppose $m-1$ coins are heads at the initial stage, determine if there is a way to carry out $\lfloor \frac{2^m}{3}\rfloor$ moves

---

5 A nonempty set A is called an [i]n-level-good [/i]set if $A\subseteq\{1,2,3,\ldots,n\}$ and $|A|\le \min_{x\in A} x$ (where $|A|$ denotes the number of elements in A and $\min_{x\in A} x$ denotes the minimum of the elements in A). Let a_n be the number of n-level-good sets. Prove that for all positive integers n we have $a_{n+2}=a_{n+1}+a_n+1$.

---

6 Let PA,PB be tangents to a circle centered at O, and C a point on the minor arc AB. The perpendicular from C to PC intersects internal angle bisectors of AOC,BOC at D,E. Show that $CD=CE$

---

7 Label sides of a regular n-gon in clockwise direction in order 1,2,..,n. Determine all integers n ($n\ge 4$) satisfying the following conditions:
(1) $n-3$ non-intersecting diagonals in the n-gon are selected, which subdivide the n-gon into $n-2$ non-overlapping triangles;
(2) each of the chosen $n-3$ diagonals are labeled with an integer, such that the sum of labeled numbers on three sides of each triangles in (1) is equal to the others;

---

8 Find all positive integers a such that for any positive integer $n\ge 5$ we have $2^n-n^2\mid a^n-n^a$.
