China National Olympiad 1986

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Day 1

1 We are given n reals $a_1, a_2, \cdots, a_n$ such that the sum of any two of them is non-negative. Prove that the following statement and its converse are both true: if n non-negative reals $x_1, x_2, \cdots, x_n$ satisfy $x_1 + x_2 + \cdots + x_n = 1$, then the inequality $a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2$ holds.

2 In $\triangle ABC$, the length of altitude AD is 12, and the bisector AE of $\angle A$ is 13. Denote by m the length of median AF. Find the range of m when $\angle A$ is acute, orthogonal and obtuse respectively.

3 Let $Z_1, Z_2, \cdots, Z_n$ be complex numbers satisfying $|Z_1| + |Z_2| + \cdots + |Z_n| = 1$. Show that there exist some among the n complex numbers such that the modulus of the sum of these complex numbers is not less than $1/6$.

Day 2

4 Given a $\triangle ABC$ with its area equal to 1, suppose that the vertices of quadrilateral $P_1P_2P_3P_4$ all lie on the sides of $\triangle ABC$. Show that among the four triangles $\triangle P_1P_2P_3, \triangle P_1P_2P_4, \triangle P_1P_3P_4, \triangle P_2P_3P_4$ there is at least one whose area is not larger than $1/4$.

5 Given a sequence $1, 1, 2, 2, 3, 3, \ldots, 1986, 1986$, determine, with proof, if we can rearrange the sequence
so that for any integer $1 \leq k \leq 1986$ there are exactly k numbers between the two ks.

6 Suppose that each point on the plane is colored either white or black. Show that there exists an equilateral triangle with the side length equal to 1 or $\sqrt{3}$ whose three vertices are in the same color.