China National Olympiad 1987

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Day 1

1 Let n be a natural number. Prove that a necessary and sufficient condition for the equation $z^{n+1} - z^n - 1 = 0$ to have a complex root whose modulus is equal to 1 is that $n+2$ is divisible by 6.

2 We are given an equilateral triangle ABC with the length of its side equal to 1. There are $n-1$ points on each side of the triangle ABC that equally divide the side into n segments. We draw all possible lines that pass through any two of all those $3(n-1)$ points such that they are parallel to one of three sides of triangle ABC. All such lines divide triangle ABC into some lesser triangles whose vertices are called *nodes*. We assign a real number for each *node* such that the following conditions are satisfied:
(I) real numbers a, b, c are assigned to A, B, C respectively;
(II) for any rhombus that is consisted of two lesser triangles that share a common side, the sum of the numbers of vertices on its one diagonal is equal to that of vertices on the other diagonal.
1) Find the minimum distance between the *node* with the maximal number to the *node* with the minimal number;
2) Denote by S the sum of the numbers of all *nodes*, find S.

3 Some players participate in a competition. Suppose that each player plays one game against every other player and there is no draw game in the competition. Player A is regarded as an excellent player if the following condition is satisfied: for any other player B, either A beats B or there exists another player C such that C beats B and A beats C. It is known that there is only one excellent player in the end, prove that this player beats all other players.

Day 2

4 Five points are arbitrarily put inside a given equilateral triangle ABC whose area is equal to 1. Show that we can draw three equilateral triangles within triangle ABC such that the following conditions are all satisfied:
i) the five points are covered by the three equilateral triangles;
ii) any side of the three equilateral triangles is parallel to a certain side of the triangle ABC;
iii) the sum of the areas of the three equilateral triangles is not larger than 0.64.

5 Let $A_1A_2A_3A_4$ be a tetrahedron. We construct four mutually tangent spheres S_1, S_2, S_3, S_4 with centers A_1, A_2, A_3, A_4 respectively. Suppose that there exists a point Q such that we can con-

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struct two spheres centered at Q satisfying the following conditions:
i) One sphere with radius r is tangent to S_1, S_2, S_3, S_4;
ii) One sphere with radius R is tangent to every edges of tetrahedron $A_1A_2A_3A_4$.
Prove that $A_1A_2A_3A_4$ is a regular tetrahedron.

6 Sum of m pairwise different positive even numbers and n pairwise different positive odd numbers is equal to 1987. Find, with proof, the maximum value of $3m+4n$.

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