China National Olympiad 1988

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Day 1

1 Let $r_1, r_2, \dots, r_n$ be real numbers. Given n reals $a_1, a_2, \dots, a_n$ that are not all equal to 0, suppose that inequality

$$r_1(x_1 - a_1) + r_2(x_2 - a_2) + \dots + r_n(x_n - a_n) \le \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} - \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

holds for arbitrary reals $x_1, x_2, \dots, x_n$. Find the values of $r_1, r_2, \dots, r_n$.

2 Given two circles C_1, C_2 with common center, the radius of C_2 is twice the radius of C_1. Quadrilateral $A_1A_2A_3A_4$ is inscribed in C_1. The extension of A_4A_1 meets C_2 at B_1; the extension of A_1A_2 meets C_2 at B_2; the extension of A_2A_3 meets C_2 at B_3; the extension of A_3A_4 meets C_2 at B_4. Prove that $P(B_1B_2B_3B_4) \ge 2P(A_1A_2A_3A_4)$, and in what case the equality holds? ($P(X)$ denotes the perimeter of quadrilateral X)

3 Given a finite sequence of real numbers $a_1, a_2, \dots, a_n$ ($\ast$), we call a segment $a_k, \dots, a_{k+l-1}$ of the sequence ($\ast$) a *long*(Chinese dragon) and a_k *head* of the *long* if the arithmetic mean of $a_k, \dots, a_{k+l-1}$ is greater than 1988. (especially if a single item $a_m > 1988$, we still regard a_m as a *long*). Suppose that there is at least one *long* among the sequence ($\ast$), show that the arithmetic mean of all those items of sequence ($\ast$) that could be *head* of a certain *long* individually is greater than 1988.

Day 2

4 (1) Let a, b, c be positive real numbers satisfying $(a^2 + b^2 + c^2)^2 > 2(a^4 + b^4 + c^4)$. Prove that a, b, c can be the lengths of three sides of a triangle respectively.
(2) Let $a_1, a_2, \dots, a_n$ be n ($n > 3$) positive real numbers satisfying $(a_1^2 + a_2^2 + \dots + a_n^2)^2 > (n-1)(a_1^4 + a_2^4 + \dots + a_n^4)$. Prove that any three of $a_1, a_2, \dots, a_n$ can be the lengths of three sides of a triangle respectively.

5 Given three tetrahedrons $A_iB_iC_iD_i$ ($i = 1, 2, 3$), planes $\alpha_i, \beta_i, \gamma_i$ ($i = 1, 2, 3$) are drawn through B_i, C_i, D_i respectively, and they are perpendicular to edges A_iB_i, A_iC_i, A_iD_i ($i = 1, 2, 3$) respectively. Suppose that all nine planes $\alpha_i, \beta_i, \gamma_i$ ($i = 1, 2, 3$) meet at a point E, and points A_1, A_2, A_3 lie on line l. Determine the intersection (shape and position) of the circumscribed spheres of the three tetrahedrons.

6 Let n ($n \ge 3$) be a natural number. Denote by $f(n)$ the least natural number by which n is not divisible (e.g. $f(12) = 5$). If $f(n) \ge 3$, we may have $f(f(n))$ in the same way. Similarly, if

$f(f(n)) \geq 3$, we may have $f(f(f(n)))$, and so on. If $\underbrace{f(f(\dots f}_{k \text{ times}}(n)\dots)) = 2$, we call k the *length* of n (also we denote by l_n the *length* of n). For arbitrary natural number n ($n \geq 3$), find l_n with proof.