China National Olympiad 1989

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Day 1

1 We are given two point sets A and B which are both composed of finite disjoint arcs on the unit circle. Moreover, the length of each arc in B is equal to $\frac{\pi}{m}$ ($m \in \mathbb{N}$). We denote by A^j the set obtained by a counterclockwise rotation of A about the center of the unit circle for $\frac{j\pi}{m}$ ($j = 1, 2, 3, \dots$). Show that there exists a natural number k such that $l(A^k \cap B) \geq \frac{1}{2\pi} l(A)l(B)$.(Here $l(X)$ denotes the sum of lengths of all disjoint arcs in the point set X)

2 Let $x_1, x_2, \dots, x_n$ ($n \geq 2$) be positive real numbers satisfying $\sum_{i=1}^n x_i = 1$. Prove that:

$$\sum_{i=1}^n \frac{x_i}{\sqrt{1-x_i}} \geq \frac{\sum_{i=1}^n \sqrt{x_i}}{\sqrt{n-1}}.$$

3 Let S be the unit circle in the complex plane (i.e. the set of all complex numbers with their moduli equal to 1).
We define function $f : S \to S$ as follow: $\forall z \in S$, $f^{(1)}(z) = f(z), f^{(2)}(z) = f(f(z)), \dots, f^{(k)}(z) = f(f^{(k-1)}(z))(k > 1, k \in \mathbb{N}), \dots$
We call c an n-*period-point* of f if c ($c \in S$) and n ($n \in \mathbb{N}$) satisfy: $f^{(1)}(c) \neq c, f^{(2)}(c) \neq c, f^{(3)}(c) \neq c, \dots, f^{(n-1)}(c) \neq c, f^{(n)}(c) = c$.
Suppose that $f(z) = z^m$ ($z \in S; m > 1, m \in \mathbb{N}$), find the number of 1989-*period-point* of f.

Day 2

4 Given a triangle ABC, points D, E, F lie on sides BC, CA, AB respectively. Moreover, the radii of incircles of $\triangle AEF, \triangle BFD, \triangle CDE$ are equal to r. Denote by r_0 and R the radii of incircles of $\triangle DEF$ and $\triangle ABC$ respectively. Prove that $r + r_0 = R$.

5 Given 1989 points in the space, any three of them are not collinear. We divide these points into 30 groups such that the numbers of points in these groups are different from each other. Consider those triangles whose vertices are points belong to three different groups among the 30. Determine the numbers of points of each group such that the number of such triangles attains a maximum.

6 Find all functions $f:(1,+\infty)\rightarrow (1,+\infty)$ that satisfy the following condition:

for arbitrary $x,y>1$ and $u,v>0$, inequality $f(x^uy^v)\le f(x)^{\dfrac{1}{4u}}f(y)^{\dfrac{1}{4v}}$ holds.