China National Olympiad 1990

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Day 1

1 Given a convex quadrilateral $ABCD$, side AB is not parallel to side CD. The circle O_1 passing through A and B is tangent to side CD at P. The circle O_2 passing through C and D is tangent to side AB at Q. Circle O_1 and circle O_2 meet at E and F. Prove that EF bisects segment PQ if and only if $BC \parallel AD$.

2 Let x be a natural number. We call $\{x_0, x_1, \dots, x_l\}$ a *factor link* of x if the sequence $\{x_0, x_1, \dots, x_l\}$ satisfies the following conditions:
(1) $x_0 = 1, x_l = x$;
(2) $x_{i-1} < x_i, x_{i-1}|x_i, i = 1, 2, \dots, l$.
Meanwhile, we define l as the length of the *factor link* $\{x_0, x_1, \dots, x_l\}$. Denote by $L(x)$ and $R(x)$ the length and the number of the longest *factor link* of x respectively. For $x = 5^k \times 31^m \times 1990^n$, where k, m, n are natural numbers, find the value of $L(x)$ and $R(x)$.

3 A function $f(x)$ defined for $x \geq 0$ satisfies the following conditions:
i. for $x, y \geq 0$, $f(x)f(y) \leq x^2 f(y/2) + y^2 f(x/2)$;
ii. there exists a constant $M(M > 0)$, such that $|f(x)| \leq M$ when $0 \leq x \leq 1$.
Prove that $f(x) \leq x^2$.

Day 2

4 Given a positive integer number a and two real numbers A and B, find a necessary and sufficient condition on A and B for the following system of equations to have integer solution:

$$\begin{cases} x^2 + y^2 + z^2 = (Ba)^2 \\ x^2(Ax^2 + By^2) + y^2(Ay^2 + Bz^2) + z^2(Az^2 + Bx^2) = \dfrac{1}{4}(2A + B)(Ba)^4 \end{cases}$$

5 Given a finite set X, let f be a rule such that f maps every *even-element-subset* E of X (i.e. $E \subseteq X$, $|E|$ is even) into a real number $f(E)$. Suppose that f satisfies the following conditions:
(I) there exists an *even-element-subset* D of X such that $f(D) > 1990$;
(II) for any two disjoint *even-element-subsets* A, B of X, equation $f(A \cup B) = f(A) + f(B) - 1990$ holds.
Prove that there exist two subsets P, Q of X satisfying:
(1) $P \cap Q = \emptyset$, $P \cup Q = X$;

(2) for any *non-even-element-subset* S of P (i.e. $S \subseteq P$, $|S|$ is odd), we have $f(S) > 1990$;
(3) for any *even-element-subset* T of Q, we have $f(T) \leq 1990$.

6 A convex n-gon and its $n-3$ diagonals which have no common point inside the polygon form a *subdivision graph*. Show that if and only if $3|n$, there exists a *subdivision graph* that can be drawn in one closed stroke. (i.e. start from a certain vertex, get through every edges and diagonals exactly one time, finally back to the starting vertex.)

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