China National Olympiad 1991

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Day 1

1 We are given a convex quadrilateral $ABCD$ in the plane.
(*i*) If there exists a point P in the plane such that the areas of $\triangle ABP, \triangle BCP, \triangle CDP, \triangle DAP$ are equal, what condition must be satisfied by the quadrilateral $ABCD$?
(*ii*) Find (with proof) the maximum possible number of such point P which satisfies the condition in (*i*).

2 Given $I = [0, 1]$ and $G = \{(x, y)|x, y \in I\}$, find all functions $f : G \to I$, such that $\forall x, y, z \in I$ we have:
i. $f(f(x, y), z) = f(x, f(y, z))$;
ii. $f(x, 1) = x, f(1, y) = y$;
iii. $f(zx, zy) = z^k f(x, y)$.
(k is a positive real number irrelevant to x, y, z.)

3 There are 10 birds on the ground. For any 5 of them, there are at least 4 birds on a circle. Determine the least possible number of birds on the circle with the most birds.

Day 2

4 Find all positive integer solutions (x, y, z, n) of equation $x^{2n+1} - y^{2n+1} = xyz + 2^{2n+1}$, where $n \geq 2$ and $z \leq 5 \times 2^{2n}$.

5 Find all natural numbers n, such that $\min_{k\in\mathbb{N}}(k^2 + [n/k^2]) = 1991$. ($[n/k^2]$ denotes the integer part of n/k^2.)

6 A football is covered by some polygonal pieces of leather which are sewed up by three different colors threads. It features as follows:
i) any edge of a polygonal piece of leather is sewed up with an equal-length edge of another polygonal piece of leather by a certain color thread;
ii) each node on the ball is vertex to exactly three polygons, and the three threads joint at the node are of different colors.
Show that we can assign to each node on the ball a complex number (not equal to 1), such that the product of the numbers assigned to the vertices of any polygonal face is equal to 1.