China National Olympiad 1992
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Day 1

1 Let equation $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0$ with real coefficients satisfy $0 < a_0 \le a_1 \le a_2 \le \cdots \le a_{n-1} \le 1$. Suppose that λ ($|\lambda| > 1$) is a complex root of the equation, prove that $\lambda^{n+1} = 1$.

2 Given nonnegative real numbers $x_1, x_2, \dots, x_n$, let $a = min\{x_1, x_2, \dots, x_n\}$. Prove that the following inequality holds:

$$\sum_{i=1}^{n} \frac{1+x_i}{1+x_{i+1}} \le n + \frac{1}{(1+a)^2}\sum_{i=1}^{n}(x_i - a)^2 \qquad (x_{n+1} = x_1),$$

and equality occurs if and only if $x_1 = x_2 = \cdots = x_n$.

3 Given a 9×9 grid, we assign either $+1$ or -1 to each square on the grid. We define an *adjustment* as follow: for each square on the grid, we make a product of all numbers of those squares which share a common side with the square (excluding itself).Then we have 81 products. Next we replace all the squares values with their corresponding products. Determine if we can make all values in the grid equal to 1 through finite *adjustments*.

Day 2

1 A convex quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC, BD of $ABCD$ meet at P. Circumcircles of $\triangle ABP$ and $\triangle CDP$ meet at P and Q (O, P, Q are pairwise distinct). Show that $\angle OQP = 90^\circ$.

2 Find the maximum possible number of edges of a simple graph with 8 vertices and without any quadrilateral. (a simple graph is an undirected graph that has no loops (edges connected at both ends to the same vertex) and no more than one edge between any two different vertices.)

3 Let sequence $\{a_1, a_2, \dots\}$ with integer terms satisfy the following conditions:
1) $a_{n+1} = 3a_n - 3a_{n-1} + a_{n-2}, n = 2, 3, \dots$;
2) $2a_1 = a_0 + a_2 - 2$;
3) for arbitrary natural number m, there exist m consecutive terms $a_k, a_{k-1}, \dots, a_{k+m-1}$ among the sequence such that all such m terms are perfect squares.
Prove that all terms of the sequence $\{a_1, a_2, \dots\}$ are perfect squares.