China National Olympiad 1993

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Day 1

1 Given an odd n, prove that there exist $2n$ integers $a_1, a_2, \cdots, a_n; b_1, b_2, \cdots, b_n$, such that for any integer k $(0 < k < n)$, the following $3n$ integers: $a_i + a_{i+1}, a_i + b_i, b_i + b_{i+k}$ $(i = 1, 2, \cdots, n; a_{n+1} = a_1, b_{n+j} = b_j, 0 < j < n)$ are of different remainders on division by $3n$.

2 Given a natural number k and a real number $a(a > 0)$, find the maximal value of $a^{k_1} + a^{k_2} + \cdots + a^{k_r}$, where $k_1 + k_2 + \cdots + k_r = k$ $(k_i \in \mathbb{N}, 1 \le r \le k)$.

3 Let K, K_1 be two circles with the same center and their radii equal to R and $R_1 (R_1 > R)$ respectively. Quadrilateral $ABCD$ is inscribed in circle K. Quadrilateral $A_1B_1C_1D_1$ is inscribed in circle K_1 where A_1, B_1, C_1, D_1 lie on rays CD, DA, AB, BC respectively. Show that $\frac{S_{A_1B_1C_1D_1}}{S_{ABCD}} \ge \frac{R_1^2}{R^2}$.

Day 2

4 We are given a set $S = \{z_1, z_2, \cdots, z_{1993}\}$, where $z_1, z_2, \cdots, z_{1993}$ are nonzero complex numbers (also viewed as nonzero vectors in the plane). Prove that we can divide S into some groups such that the following conditions are satisfied:
(1) Each element in S belongs and only belongs to one group;
(2) For any group p, if we use $T(p)$ to denote the sum of all memebers in p, then for any memeber $z_i (1 \le i \le 1993)$ of p, the angle between z_i and $T(p)$ does not exceed 90°;
(3) For any two groups p and q, the angle between $T(p)$ and $T(q)$ exceeds 90° (use the notation introduced in (2)).

5 10 students bought some books in a bookstore. It is known that every student bought exactly three kinds of books, and any two of them shared at least one kind of book. Determine, with proof, how many students bought the most popular book at least? (Note: the most popular book means most students bought this kind of book)

6 Let $f : (0, +\infty) \to (0, +\infty)$ be a function satisfying the following condition: for arbitrary positive real numbers x and y, we have $f(xy) \le f(x)f(y)$. Show that for arbitrary positive real number x and natural number n, inequality $f(x^n) \le f(x)f(x^2)^{\frac{1}{2}} \dots f(x^n)^{\frac{1}{n}}$ holds.