China National Olympiad 1994

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Day 1

1 Let $ABCD$ be a trapezoid with $AB \parallel CD$. Points E, F lie on segments AB, CD respectively. Segments CE, BF meet at H, and segments ED, AF meet at G. Show that $S_{EHFG} \le \frac{1}{4} S_{ABCD}$. Determine, with proof, if the conclusion still holds when $ABCD$ is just any convex quadrilateral.

2 There are m pieces of candy held in n trays($n, m \ge 4$). An *operation* is defined as follow: take out one piece of candy from any two trays respectively, then put them in a third tray. Determine, with proof, if we can move all candies to a single tray by finite *operations*.

3 Find all functions $f : [1, \infty) \to [1, \infty)$ satisfying the following conditions:
(1) $f(x) \le 2(x + 1)$;
(2) $f(x + 1) = \frac{1}{x}[(f(x))^2 - 1]$.

Day 2

4 Let $f(z) = c_0 z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_{n-1} z + c_n$ be a polynomial with complex coefficients. Prove that there exists a complex number z_0 such that $|f(z_0)| \ge |c_0| + |c_n|$, where $|z_0| \le 1$.

5 For arbitrary natural number n, prove that $\sum_{k=0}^{n} C_n^k 2^k C_{n-k}^{[(n-k)/2]} = C_{2n+1}^n$, where $C_0^0 = 1$ and $[\frac{n-k}{2}]$ denotes the integer part of $\frac{n-k}{2}$.

6 Let M be a point which has coordinates $(p \times 1994, 7p \times 1994)$ in the Cartesian plane (p is a prime). Find the number of right-triangles satisfying the following conditions:
(1) all vertexes of the triangle are lattice points, moreover M is on the right-angled corner of the triangle;
(2) the origin $(0, 0)$ is the incenter of the triangle.