China National Olympiad 1995

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Day 1

1 Let $a_1, a_2, \cdots, a_n; b_1, b_2, \cdots, b_n (n \geq 3)$ be real numbers satisfying the following conditions:
(1) $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$;
(2) $0 < a_1 = a_2, a_i + a_{i+1} = a_{i+2}$ $(i = 1, 2, \cdots, n-2)$;
(3) $0 < b_1 \leq b_2, b_i + b_{i+1} \leq b_{i+2}$ $(i = 1, 2, \cdots, n-2)$.
Prove that $a_{n-1} + a_n \leq b_{n-1} + b_n$.

2 Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying the following conditions:
(1) $f(1) = 1$;
(2) $\forall n \in \mathbb{N}$, $3f(n)f(2n+1) = f(2n)(1 + 3f(n))$;
(3) $\forall n \in \mathbb{N}$, $f(2n) < 6f(n)$.
Find all solutions of equation $f(k) + f(l) = 293$, where $k < l$.
($\mathbb{N}$ denotes the set of all natural numbers).

3 Find the minimun value of $\sum_{i=1}^{10} \sum_{j=1}^{10} \sum_{k=1}^{10} |k(x+y-10i)(3x-6y-36j)(19x+95y-95k)|$, where x, y are integers.

Day 2

1 Given four spheres with their radii equal to $2, 2, 3, 3$ respectively, each sphere externally touches the other spheres. Suppose that there is another sphere that is externally tangent to all those four spheres, determine the radius of this sphere.

2 Let $a_1, a_2, \cdots, a_{10}$ be pairwise distinct natural numbers with their sum equal to 1995. Find the minimal value of $a_1a_2 + a_2a_3 + \cdots + a_9a_{10} + a_{10}a_1$.

3 Let $n (n > 1)$ be an odd. We define $x_k = (x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)})$ as follow: $x_0 = (x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}) = (1, 0, \cdots, 0, 1)$; $x_i^{(k)} = \begin{cases} 0, & x_i^{(k-1)} = x_{i+1}^{(k-1)}, \\ 1, & x_i^{(k-1)} \neq x_{i+1}^{(k-1)}, \end{cases}$ $i = 1, 2, \cdots, n$, where $x_{n+1}^{(k-1)} = x_1^{(k-1)}$.
Let m be a positive integer satisfying $x_0 = x_m$. Prove that m is divisible by n.