China National Olympiad 1996

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Day 1

1 Let $\triangle ABC$ be a triangle with orthocentre H. The tangent lines from A to the circle with diameter BC touch this circle at P and Q. Prove that H, P and Q are collinear.

2 Find the smallest positive integer K such that every K-element subset of $\{1, 2, ..., 50\}$ contains two distinct elements a, b such that $a + b$ divides ab.

3 Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x^3 + y^3) = (x + y)(f(x)^2 - f(x)f(y) + f(y)^2)$$

for all $x, y \in \mathbb{R}$.
Prove that $f(1996x) = 1996f(x)$ for all $x \in \mathbb{R}$.

Day 2

1 8 singers take part in a festival. The organiser wants to plan m concerts. For every concert there are 4 singers who go on stage, with the restriction that the times of which every two singers go on stage in a concert are all equal. Find a schedule that minimises m.

2 Let n be a natural number. Suppose that $x_0 = 0$ and that $x_i > 0$ for all $i \in \{1, 2, \ldots, n\}$. If $\sum_{i=1}^n x_i = 1$, prove that

$$1 \leq \sum_{i=1}^{n} \frac{x_i}{\sqrt{1 + x_0 + x_1 + \ldots + x_{i-1}}\sqrt{x_i + \ldots + x_n}} < \frac{\pi}{2}$$

3 In the triangle ABC, $\angle C = 90^\circ, \angle A = 30^\circ$ and $BC = 1$. Find the minimum value of the longest side of all inscribed triangles (i.e. triangles with vertices on each of three sides) of the triangle ABC.