China National Olympiad 1997

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Day 1

1 Let $x_1, x_2, \ldots, x_{1997}$ be real numbers satisfying the following conditions:
i) $-\frac{1}{\sqrt{3}} \le x_i \le \sqrt{3}$ for $i = 1, 2, \ldots, 1997$;
ii) $x_1 + x_2 + \cdots + x_{1997} = -318\sqrt{3}$.
Determine (with proof) the maximum value of $x_1^{12} + x_2^{12} + \ldots + x_{1997}^{12}$.

2 Let $A_1B_1C_1D_1$ be an arbitrary convex quadrilateral. P is a point inside the quadrilateral such that each angle enclosed by one edge and one ray which starts at one vertex on that edge and passes through point P is acute. We recursively define points A_k, B_k, C_k, D_k symmetric to P with respect to lines $A_{k-1}B_{k-1}, B_{k-1}C_{k-1}, C_{k-1}D_{k-1}, D_{k-1}A_{k-1}$ respectively for $k \ge 2$.
Consider the sequence of quadrilaterals $A_iB_iC_iD_i$.
i) Among the first 12 quadrilaterals, which are similar to the 1997th quadrilateral and which are not?
ii) Suppose the 1997th quadrilateral is cyclic. Among the first 12 quadrilaterals, which are cyclic and which are not?

3 Prove that there are infinitely many natural numbers n such that we can divide $1, 2, \ldots, 3n$ into three sequences $(a_n), (b_n)$ and (c_n), with n terms in each, satisfying the following conditions:
i) $a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = \ldots = a_n + b_n + c_n$ and $a_1 + b_1 + c_1$ is divisible by 6;
ii) $a_1 + a_2 + \ldots + a_n = b_1 + b_2 + \ldots + b_n = c_1 + c_2 + \ldots + c_n$, and $a_1 + a_2 + \ldots + a_n$ is divisible by 6.

Day 2

1 Consider a cyclic quadrilateral $ABCD$. The extensions of its sides AB, DC meet at the point P and the extensions of its sides AD, BC meet at the point Q. Suppose QE, QF are tangents to the circumcircle of quadrilateral $ABCD$ at E, F respectively. Show that P, E, F are collinear.

2 Let $A = \{1, 2, 3, \cdots, 17\}$. A mapping $f : A \to A$ is defined as follows: $f^{[1]}(x) = f(x), f^{[k+1]}(x) = f(f^{[k]}(x))$ for $k \in \mathbb{N}$. Suppose that f is bijective and that there exists a natural number M such that:
i) when $m < M$ and $1 \le i \le 16$, we have $f^{[m]}(i+1) - f^{[m]}(i) \ne \pm 1 \pmod{17}$ and $f^{[m]}(1) - f^{[m]}(17) \ne \pm 1 \pmod{17}$;
ii) when $1 \le i \le 16$, we have $f^{[M]}(i+1) - f^{[M]}(i) = \pm 1 \pmod{17}$ and $f^{[M]}(1) - f^{[M]}(17) = \pm 1$

$\pmod{17}$.
Find the maximal value of M.

3 Let (a_n) be a sequence of non-negative real numbers satisfying $a_{n+m} \leq a_n + a_m$ for all non-negative integers m, n.
Prove that if $n \geq m$ then $a_n \leq ma_1 + \left(\frac{n}{m} - 1\right) a_m$ holds.