China National Olympiad 1998

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Day 1

1 Let ABC be a non-obtuse triangle satisfying $AB > AC$ and $\angle B = 45^\circ$. The circumcentre O and incentre I of triangle ABC are such that $\sqrt{2}\,OI = AB - AC$. Find the value of $\sin A$.

2 Given a positive integer $n > 1$, determine with proof if there exist $2n$ pairwise different positive integers $a_1, \ldots, a_n, b_1, \ldots b_n$ such that $a_1 + \ldots + a_n = b_1 + \ldots + b_n$ and

$$n - 1 > \sum_{i=1}^{n} \frac{a_i - b_i}{a_i + b_i} > n - 1 - \frac{1}{1998}.$$

3 Let $S = \{1, 2, \ldots, 98\}$. Find the least natural number n such that we can pick out 10 numbers in any n-element subset of S satisfying the following condition: no matter how we equally divide the 10 numbers into two groups, there exists a number in one group such that it is coprime to the other numbers in that group, meanwhile there also exists a number in the other group such that it is not coprime to any of the other numbers in the same group.

Day 2

1 Find all natural numbers $n > 3$, such that 2^{2000} is divisible by $1 + C_n^1 + C_n^2 + C_n^3$.

2 Let D be a point inside acute triangle ABC satisfying the condition

$$DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA = AB \cdot BC \cdot CA.$$

Determine (with proof) the geometric position of point D.

3 Let $x_1, x_2, \ldots, x_n$ be real numbers, where $n \geq 2$, satisfying $\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n-1} x_i x_{i+1} = 1$. For each k, find the maximal value of $|x_k|$.