**China National Olympiad 1999**

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**Day 1**

**1** Let $ABC$ be an acute triangle with $\angle C > \angle B$. Let $D$ be a point on $BC$ such that $\angle ADB$ is obtuse, and let $H$ be the orthocentre of triangle $ABD$. Suppose that $F$ is a point inside triangle $ABC$ that is on the circumcircle of triangle $ABD$. Prove that $F$ is the orthocenter of triangle $ABC$ if and only if $HD||CF$ and $H$ is on the circumcircle of triangle $ABC$.

**2** Let $a$ be a real number. Let $(f_n(x))_{n\ge 0}$ be a sequence of polynomials such that $f_0(x)=1$ and $f_{n+1}(x)=xf_n(x)+f_n(ax)$ for all non-negative integers $n$.
a) Prove that $f_n(x)=x^nf_n\left(x^{-1}\right)$ for all non-negative integers $n$.
b) Find an explicit expression for $f_n(x)$.

**3** There are $99$ space stations. Each pair of space stations is connected by a tunnel. There are $99$ two-way main tunnels, and all the other tunnels are strictly one-way tunnels. A group of $4$ space stations is called *connected* if one can reach each station in the group from every other station in the group without using any tunnels other than the $6$ tunnels which connect them. Determine the maximum number of connected groups.

**Day 2**

**1** Let $m$ be a positive integer. Prove that there are integers $a, b, k$, such that both $a$ and $b$ are odd, $k\ge 0$ and

$$2m=a^{19}+b^{99}+k\cdot 2^{1999}$$

**2** Determine the maximum value of $\lambda$ such that if $f(x)=x^3+ax^2+bx+c$ is a cubic polynomial with all its roots nonnegative, then

$$f(x)\ge\lambda(x-a)^3$$

for all $x\ge 0$. Find the equality condition.

**3** A $4\times 4\times 4$ cube is composed of $64$ unit cubes. The faces of $16$ unit cubes are to be coloured red. A colouring is called interesting if there is exactly $1$ red unit cube in every $1\times 1\times 4$ rectangular box composed of $4$ unit cubes. Determine the number of interesting colourings.