China National Olympiad 2000

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Day 1

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1 The sides a, b, c of triangle ABC satisfy $a \leq b \leq c$. The circumradius and inradius of triangle ABC are R and r respectively. Let $f = a + b - 2R - 2r$. Determine the sign of f by the measure of angle C.

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2 A sequence (a_n) is defined recursively by $a_1 = 0, a_2 = 1$ and for $n \geq 3$,

$$a_n = \frac{1}{2}na_{n-1} + \frac{1}{2}n(n-1)a_{n-2} + (-1)^n\left(1 - \frac{n}{2}\right).$$

Find a closed-form expression for $f_n = a_n + 2\binom{n}{1}a_{n-1} + 3\binom{n}{2}a_{n-2} + \ldots + (n-1)\binom{n}{n-2}a_2 + n\binom{n}{n-1}a_1$.

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3 A table tennis club hosts a series of doubles matches following several rules:
(i) each player belongs to two pairs at most;
(ii) every two distinct pairs play one game against each other at most;
(iii) players in the same pair do not play against each other when they pair with others respectively.
Every player plays a certain number of games in this series. All these distinct numbers make up a set called the *set of games*. Consider a set $A = \{a_1, a_2, \ldots, a_k\}$ of positive integers such that every element in A is divisible by 6. Determine the minimum number of players needed to participate in this series so that a schedule for which the corresponding *set of games* is equal to set A exists.

Day 2

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1 Given an ordered n-tuple $A = (a_1, a_2, \cdots, a_n)$ of real numbers, where $n \geq 2$, we define $b_k = \max a_1, \ldots a_k$ for each k. We define $B = (b_1, b_2, \cdots, b_n)$ to be the *innovated tuple* of A. The number of distinct elements in B is called the *innovated degree* of A.
Consider all permutations of $1, 2, \ldots, n$ as an ordered n-tuple. Find the arithmetic mean of the first term of the permutations whose innovated degrees are all equal to 2

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2 Find all positive integers n such that there exists integers $n_1, \ldots, n_k \geq 3$, for some integer k, satisfying

$$n = n_1 n_2 \cdots n_k = 2^{\frac{1}{2^k}(n_1-1)\cdots(n_k-1)} - 1.$$

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3 A test contains 5 multiple choice questions which have 4 options in each. Suppose each examinee chose one option for each question. There exists a number n, such that for any n sheets among 2000 sheets of answer papers, there are 4 sheets of answer papers such that any two of them have at most 3 questions with the same answers. Find the minimum value of n.