China National Olympiad 2001
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by horizon

Day 1

1 Let a be real number with $\sqrt{2} < a < 2$, and let $ABCD$ be a convex cyclic quadrilateral whose circumcentre O lies in its interior. The quadrilateral's circumcircle ω has radius 1, and the longest and shortest sides of the quadrilateral have length a and $\sqrt{4-a^2}$, respectively. Lines L_A, L_B, L_C, L_D are tangent to ω at A, B, C, D, respectively.

Let lines L_A and L_B, L_B and L_C,L_C and L_D,L_D and L_A intersect at A', B', C', D' respectively. Determine the minimum value of $\frac{S_{A'B'C'D'}}{S_{ABCD}}$.

2 Let $X = \{1, 2, \dots, 2001\}$. Find the least positive integer m such that for each subset $W \subset X$ with m elements, there exist $u, v \in W$ (not necessarily distinct) such that $u + v$ is of the form 2^k, where k is a positive integer.

3 Let P be a regular n-gon $A_1A_2\dots A_n$. Find all positive integers n such that for each permutation $\sigma(1), \sigma(2), \dots, \sigma(n)$ there exists $1 \le i, j, k \le n$ such that the triangles $A_iA_jA_k$ and $A_{\sigma(i)}A_{\sigma(j)}A_{\sigma(k)}$ are both acute, both right or both obtuse.

Day 2

1 Let a, b, c be positive integers such that $a, b, c, a+b-c, a+c-b, b+c-a, a+b+c$ are 7 distinct primes. The sum of two of a, b, c is 800. If d be the difference of the largest prime and the least prime among those 7 primes, find the maximum value of d.

2 Let $P_1P_2\dots P_{24}$ be a regular 24-sided polygon inscribed in a circle ω with circumference 24. Determine the number of ways to choose sets of eight distinct vertices from these 24 such that none of the arcs has length 3 or 8.

3 Let $a = 2001$. Consider the set A of all pairs of integers (m, n) with $n \neq 0$ such that
(i) $m < 2a$;
(ii) $2n|(2am - m^2 + n^2)$;
(iii) $n^2 - m^2 + 2mn \le 2a(n-m)$.
For $(m, n) \in A$, let

$$f(m,n) = \frac{2am - m^2 - mn}{n}.$$

Determine the maximum and minimum values of f.