China National Olympiad 2003

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by littletush

Day 1 January 15th

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1 Let I and H be the incentre and orthocentre of triangle ABC respectively. Let P, Q be the midpoints of AB, AC. The rays PI, QI intersect AC, AB at R, S respectively. Suppose that T is the circumcentre of triangle BHC. Let RS intersect BC at K. Prove that A, I and T are collinear if and only if $[BKS] = [CKR]$.

Shen Wunxuan

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2 Determine the maximal size of the set S such that:
i) all elements of S are natural numbers not exceeding 100;
ii) for any two elements a, b in S, there exists c in S such that $(a, c) = (b, c) = 1$;
iii) for any two elements a, b in S, there exists d in S such that $(a, d) > 1, (b, d) > 1$.

Yao Jiangang

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3 Given a positive integer n, find the least $\lambda > 0$ such that for any $x_1, \dots x_n \in \left(0, \frac{\pi}{2}\right)$, the condition $\prod_{i=1}^{n} \tan x_i = 2^{\frac{n}{2}}$ implies $\sum_{i=1}^{n} \cos x_i \le \lambda$.

Huang Yumin

Day 2 January 16th

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1 Find all integer triples (a, m, n) such that $a^m + 1 | a^n + 203$ where $a, m > 1$.

Chen Yonggao

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2 Ten people apply for a job. The manager decides to interview the candidates one by one according to the following conditions:
i) the first three candidates will not be employed;
ii) from the fourth candidates onwards, if a candidate's comptence surpasses the competence of all those who preceded him, then that candidate is employed;
iii) if the first nine candidates are not employed, then the tenth candidate will be employed.
We assume that none of the 10 applicants have the same competence, and these competences can be ranked from the first to tenth. Let P_k represent the probability that the kth-ranked applicant in competence is employed. Prove that:
i) $P_1 > P_2 > \dots > P_8 = P_9 = P_{10}$;
ii) $P_1 + P_2 + P_3 > 0.7$
iii) $P_8 + P_9 + P_{10} \le 0.1$.

Su Chun

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3 Suppose a, b, c, d are positive reals such that $ab + cd = 1$ and x_i, y_i are real numbers such that $x_i^2 + y_i^2 = 1$ for $i = 1, 2, 3, 4$. Prove that

$$(ax_1 + bx_2 + cx_3 + dx_4)^2 + (ay_4 + by_3 + cy_2 + dy_1)^2 \leq 2\left(\frac{a^2 + b^2}{ab} + \frac{c^2 + d^2}{cd}\right).$$

Li Shenghong
