China National Olympiad 2005

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Day 1

1 Suppose $\theta_i\in(-\frac{\pi}{2},\frac{\pi}{2}), i=1,2,3,4$. Prove that, there exist $x\in\mathbb{R}$, satisfying two inequalities

$$\begin{array}{rcl}\cos^2\theta_1\cos^2\theta_2-(\sin\theta\sin\theta_2-x)^2 & \geq & 0,\\ \cos^2\theta_3\cos^2\theta_4-(\sin\theta_3\sin\theta_4-x)^2 & \geq & 0\end{array}$$

if and only if

$$\sum_{i=1}^4\sin^2\theta_i\leq 2(1+\prod_{i=1}^4\sin\theta_i+\prod_{i=1}^4\cos\theta_i).$$

2 A circle meets the three sides BC,CA,AB of a triangle ABC at points $D_1,D_2;E_1,E_2;F_1,F_2$ respectively. Furthermore, line segments D_1E_1 and D_2F_2 intersect at point L, line segments E_1F_1 and E_2D_2 intersect at point M, line segments F_1D_1 and F_2E_2 intersect at point N. Prove that the lines AL,BM,CN are concurrent.

3 As the graph, a pond is divided into 2n (n $\geq$ 5) parts. Two parts are called neighborhood if they have a common side or arc. Thus every part has three neighborhoods. Now there are 4n+1 frogs at the pond. If there are three or more frogs at one part, then three of the frogs of the part will jump to the three neighborhoods repsectively. Prove that for some time later, the frogs at the pond will uniformily distribute. That is, for any part either there are frogs at the part or there are frogs at the each of its neighborhoods.

`http://www.mathlinks.ro/Forum/files/china2005_2_214.gif`

Day 2

4 The sequence $\{a_n\}$ is defined by: $a_1=\frac{21}{16}$, and for $n\geq 2$,

$$2a_n-3a_{n-1}=\frac{3}{2^{n+1}}.$$

Let m be an integer with $m\geq 2$. Prove that: for $n\leq m$, we have

$$\left(a_n+\frac{3}{2^{n+3}}\right)^{\frac{1}{m}}\left(m-\left(\frac{2}{3}\right)^{\frac{n(m-1)}{m}}\right)<\frac{m^2-1}{m-n+1}.$$

5 There are 5 points in a rectangle (including its boundary) with area 1, no three of them are in the same line. Find the minimum number of triangles with the area not more than $\frac{1}{4}$, vertex of which are three of the five points.

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6 Find all nonnegative integer solutions (x, y, z, w) of the equation

$$2^x \cdot 3^y - 5^z \cdot 7^w = 1.$$
