China National Olympiad 2006

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Day 1 January 12th

1 Let $a_1, a_2, \ldots, a_k$ be real numbers and $a_1 + a_2 + \ldots + a_k = 0$. Prove that

$$\max_{1 \le i \le k} a_i^2 \le \frac{k}{3}\left((a_1 - a_2)^2 + (a_2 - a_3)^2 + \cdots + (a_{k-1} - a_k)^2\right).$$

2 For positive integers $a_1, a_2, \ldots, a_{2006}$ such that $\frac{a_1}{a_2}, \frac{a_2}{a_3}, \ldots, \frac{a_{2005}}{a_{2006}}$ are pairwise distinct, find the minimum possible amount of distinct positive integers in the set$\{a_1, a_2, ..., a_{2006}\}$.

3 Positive integers k, m, n satisfy $mn = k^2 + k + 3$, prove that at least one of the equations $x^2 + 11y^2 = 4m$ and $x^2 + 11y^2 = 4n$ has an odd solution.

Day 2 January 13th

4 In a right angled-triangle ABC, $\angle ACB = 90^o$. Its incircle O meets BC, AC, AB at D,E,F respectively. AD cuts O at P. If $\angle BPC = 90^o$, prove $AE + AP = PD$.

5 Let $\{a_n\}$ be a sequence such that: $a_1 = \frac{1}{2}$, $a_{k+1} = -a_k + \frac{1}{2 - a_k}$ for all $k = 1, 2, \ldots$. Prove that

$$\left(\frac{n}{2(a_1 + a_2 + \cdots + a_n)} - 1\right)^n \le \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n \left(\frac{1}{a_1} - 1\right)\left(\frac{1}{a_2} - 1\right)\cdots\left(\frac{1}{a_n} - 1\right).$$

6 Suppose X is a set with $|X| = 56$. Find the minimum value of n, so that for any 15 subsets of X, if the cardinality of the union of any 7 of them is greater or equal to n, then there exists 3 of them whose intersection is nonempty.