**China National Olympiad 2007**
www.artofproblemsolving.com/community/c5230
by Lei Lei

**Day 1**

**1** Given complex numbers $a, b, c$, let $|a+b|=m, |a-b|=n$. If $mn\neq 0$, Show that

$$\max\{|ac+b|,|a+bc|\}\geq\frac{mn}{\sqrt{m^2+n^2}}$$

**2** Show that:
1) If $2n-1$ is a prime number, then for any $n$ pairwise distinct positive integers $a_1, a_2, \dots, a_n$, there exists $i, j\in\{1,2,\dots,n\}$ such that

$$\frac{a_i+a_j}{(a_i,a_j)}\geq 2n-1$$

2) If $2n-1$ is a composite number, then there exists $n$ pairwise distinct positive integers $a_1, a_2, \dots, a_n$, such that for any $i, j\in\{1,2,\dots,n\}$ we have

$$\frac{a_i+a_j}{(a_i,a_j)}<2n-1$$

Here $(x, y)$ denotes the greatest common divisor of $x, y$.

**3** Let $a_1, a_2, \dots, a_{11}$ be 11 pairwise distinct positive integer with sum less than 2007. Let S be the sequence of $1, 2, \dots, 2007$. Define an **operation** to be 22 consecutive applications of the following steps on the sequence $S$: on $i$-th step, choose a number from the sequense $S$ at random, say $x$. If $1\leq i\leq 11$, replace $x$ with $x+a_i$ ; if $12\leq i\leq 22$, replace $x$ with $x-a_{i-11}$ . If the result of **operation** on the sequence $S$ is an odd permutation of $\{1, 2, \dots, 2007\}$, it is an **odd operation**; if the result of **operation** on the sequence $S$ is an even permutation of $\{1, 2, \dots, 2007\}$, it is an **even operation**. Which is larger, the number of odd operation or the number of even permutation? And by how many?

Here $\{x_1, x_2, \dots, x_{2007}\}$ is an even permutation of $\{1, 2, \dots, 2007\}$ if the product $\prod_{i>j}(x_i-x_j)$ is positive, and an odd one otherwise.

**Day 2**

**1** Let $O, I$ be the circumcenter and incenter of triangle $ABC$. The incircle of $\triangle ABC$ touches $BC, CA, AB$ at points $D, E, F$ repsectively. $FD$ meets $CA$ at $P$, $ED$ meets $AB$ at $Q$. $M$ and $N$ are midpoints of $PE$ and $QF$ respectively. Show that $OI\perp MN$.

**2** Let $\{a_n\}_{n\geq 1}$ be a bounded sequence satisfying

$$a_n < \sum_{k=a}^{2n+2006} \frac{a_k}{k+1} + \frac{1}{2n+2007} \quad \forall \quad n = 1, 2, 3, \ldots$$

Show that

$$a_n < \frac{1}{n} \quad \forall \quad n = 1, 2, 3, \ldots$$

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**3** Find a number $n \geq 9$ such that for any $n$ numbers, not necessarily distinct, $a_1, a_2, \ldots, a_n$, there exists 9 numbers $a_{i_1}, a_{i_2}, \ldots, a_{i_9}, (1 \leq i_1 < i_2 < \ldots < i_9 \leq n)$ and $b_i \in 4, 7, i = 1, 2, \ldots, 9$ such that $b_1 a_{i_1} + b_2 a_{i_2} + \ldots + b_9 a_{i_9}$ is a multiple of 9.

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