China National Olympiad 2008

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Day 1

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1 Suppose $\triangle ABC$ is scalene. O is the circumcenter and A' is a point on the extension of segment AO such that $\angle BA'A = \angle CA'A$. Let point A_1 and A_2 be foot of perpendicular from A' onto AB and AC. H_A is the foot of perpendicular from A onto BC. Denote R_A to be the radius of circumcircle of $\triangle H_A A_1 A_2$. Similiarly we can define R_B and R_C. Show that:

$$\frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C}=\frac{2}{R}$$

where R is the radius of circumcircle of $\triangle ABC$.

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2 Given an integer $n\geq 3$, prove that the set $X=\{1,2,3,\ldots,n^2-n\}$ can be divided into two non-intersecting subsets such that neither of them contains n elements $a_1,a_2,\ldots,a_n$ with $a_1<a_2<\ldots<a_n$ and $a_k\leq\frac{a_{k-1}+a_{k+1}}{2}$ for all $k=2,\ldots,n-1$.

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3 Given a positive integer n and $x_1\leq x_2\leq\ldots\leq x_n, y_1\geq y_2\geq\ldots\geq y_n$, satisfying

$$\sum_{i=1}^{n} ix_i=\sum_{i=1}^{n} iy_i$$

Show that for any real number α, we have

$$\sum_{i=1}^{n} x_i[i\alpha]\geq\sum_{i=1}^{n} y_i[i\alpha]$$

Here $[\beta]$ denotes the greastest integer not larger than β.

Day 2

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1 Let A be an infinite subset of $\mathbb{N}$, and n a fixed integer. For any prime p not dividing n, There are infinitely many elements of A not divisible by p. Show that for any integer $m>1, (m,n)=1$, There exist finitely many elements of A, such that their sum is congruent to 1 modulo m and congruent to 0 modulo n.

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2 Find the smallest integer n satisfying the following condition: regardless of how one colour the vertices of a regular n-gon with either red, yellow or blue, one can always find an isosceles trapezoid whose vertices are of the same colour.

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3 Find all triples (p,q,n) that satisfy

$$q^{n+2} \equiv 3^{n+2} (\mod p^n), \quad p^{n+2} \equiv 3^{n+2} (\mod q^n)$$

where p,q are odd primes and n is an positive integer.