China National Olympiad 2010

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Day 1

1 Two circles Γ_1 and Γ_2 meet at A and B. A line through B meets Γ_1 and Γ_2 again at C and D repsectively. Another line through B meets Γ_1 and Γ_2 again at E and F repsectively. Line CF meets Γ_1 and Γ_2 again at P and Q respectively. M and N are midpoints of arc PB and arc QB repsectively. Show that if $CD = EF$, then C, F, M, N are concyclic.

2 Let k be an integer ≥ 3. Sequence $\{a_n\}$ satisfies that $a_k = 2k$ and for all $n > k$, we have

$$a_n = \begin{cases} a_{n-1}+1 & \text{if } (a_{n-1}, n) = 1 \\ 2n & \text{if } (a_{n-1}, n) > 1 \end{cases}$$

Prove that there are infinitely many primes in the sequence $\{a_n - a_{n-1}\}$.

3 Given complex numbers a, b, c, we have that $|az^2 + bz + c| \leq 1$ holds true for any complex number $z, |z| \leq 1$. Find the maximum value of $|bc|$.

Day 2

1 Let $m, n \geq 1$ and $a_1 < a_2 < \ldots < a_n$ be integers. Prove that there exists a subset T of $\mathbb{N}$ such that

$$|T| \leq 1 + \frac{a_n - a_1}{2n+1}$$

and for every $i \in \{1, 2, \ldots, m\}$, there exists $t \in T$ and $s \in [-n, n]$, such that $a_i = t + s$.

2 There is a deck of cards placed at every points $A_1, A_2, \ldots, A_n$ and O, where $n \geq 3$. We can do one of the following two operations at each step: $1)$ If there are more than 2 cards at some points A_i, we can withdraw three cards from that deck and place one each at A_{i-1}, A_{i+1} and O. (Here $A_0 = A_n$ and $A_{n+1} = A_1$); $2)$ If there are more than or equal to n cards at point O, we can withdraw n cards from that deck and place one each at $A_1, A_2, \ldots, A_n$.
Show that if the total number of cards is more than or equal to $n^2 + 3n + 1$, we can make the number of cards at every points more than or equal to $n + 1$ after finitely many steps.

3 Suppose $a_1, a_2, a_3, b_1, b_2, b_3$ are distinct positive integers such that

$$(n+1)a_1^n + na_2^n + (n-1)a_3^n | (n+1)b_1^n + nb_2^n + (n-1)b_3^n$$

holds for all positive integers n. Prove that there exists $k \in N$ such that $b_i = ka_i$ for $i = 1, 2, 3$.