Spain Mathematical Olympiad 2013

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1 Let a, b, n positive integers with $a > b$ and $ab - 1 = n^2$. Prove that $a - b \geq \sqrt{4n - 3}$ and study the cases where the equality holds.

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2 Find all the possible values of a positive integer n for which the expression $S_n = x^n + y^n + z^n$ is constant for all real x, y, z with $xyz = 1$ and $x + y + z = 0$.

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3 Let k, n be positive integers with $n \geq k \geq 3$. We consider $n + 1$ points on the real plane with none three of them on the same line. We colour any segment between the points with one of k possibilities. We say that an angle is a "bicolour angle" iff its vertex is one of the $n+1$ points and the two segments that define it are of different colours. Show that there is always a way to colour the segments that makes more than $n\left\lfloor\frac{n}{k}\right\rfloor^2\frac{k(k-1)}{2}$ bicolour angles.

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4 Are there infinitely many positive integers n that can not be represented as $n = a^3 + b^5 + c^7 + d^9 + e^{11}$, where a, b, c, d, e are positive integers? Explain why.

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5 Study if it there exist an strictly increasing sequence of integers $0 = a_0 < a_1 < a_2 < ...$ satisfying the following conditions

$i)$ Any natural number can be written as the sum of two terms of the sequence (not necessarily distinct).

$ii)$For any positive integer n we have $a_n > \frac{n^2}{16}$

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6 Let $ABCD$ a convex quadrilateral where:

$|AB| + |CD| = \sqrt{2}|AC|$ and $|BC| + |DA| = \sqrt{2}|BD|$

What form does the quadrilateral have?
