China National Olympiad 2011

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Day 1

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1 Let $a_1, a_2, \ldots, a_n$ are real numbers, prove that;

$$\sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} a_i a_{i+1} \le \left\lfloor \frac{n}{2} \right\rfloor (M-m)^2.$$

where $a_{n+1} = a_1, M = \max_{1 \le i \le n} a_i, m = \min_{1 \le i \le n} a_i$.

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2 On the circumcircle of the acute triangle ABC, D is the midpoint of $\widehat{BC}$. Let X be a point on $\widehat{BD}$, E the midpoint of $\widehat{AX}$, and let S lie on $\widehat{AC}$. The lines SD and BC have intersection R, and the lines SE and AX have intersection T. If $RT \parallel DE$, prove that the incenter of the triangle ABC is on the line RT.

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3 Let A be a set consist of finite real numbers,$A_1, A_2, \cdots, A_n$ be nonempty sets of A, such that

(a) The sum of the elements of A is 0,

(b) For all $x_i \in A_i (i = 1, 2, \cdots, n)$,we have $x_1 + x_2 + \cdots + x_n > 0$.

Prove that there exist $1 \le k \le n$, and $1 \le i_1 < i_2 < \cdots < i_k \le n$, such that

$$|A_{i_1} \bigcup A_{i_2} \bigcup \cdots \bigcup A_{i_k}| < \frac{k}{n}|A|.$$

Where $|X|$ denote the numbers of the elements in set X.

Day 2

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1 Let n be an given positive integer, the set $S = \{1, 2, \cdots, n\}$.For any nonempty set A and B, find the minimum of $|A\Delta S| + |B\Delta S| + |C\Delta S|$, where $C = \{a+b | a \in A, b \in B\}, X\Delta Y = X \cup Y - X \cap Y$.

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2 Let $a_i, b_i, i = 1, \cdots, n$ are nonnegitive numbers,and $n \ge 4$,such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n > 0$.

Find the maximum of $\frac{\sum_{i=1}^{n} a_i(a_i+b_i)}{\sum_{i=1}^{n} b_i(a_i+b_i)}$

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3 Let m, n be positive integer numbers. Prove that there exist infinite many couples of positive integer nubmers (a, b) such that

$$a + b | am^a + bn^b, \quad \gcd(a, b) = 1.$$