China National Olympiad 2012

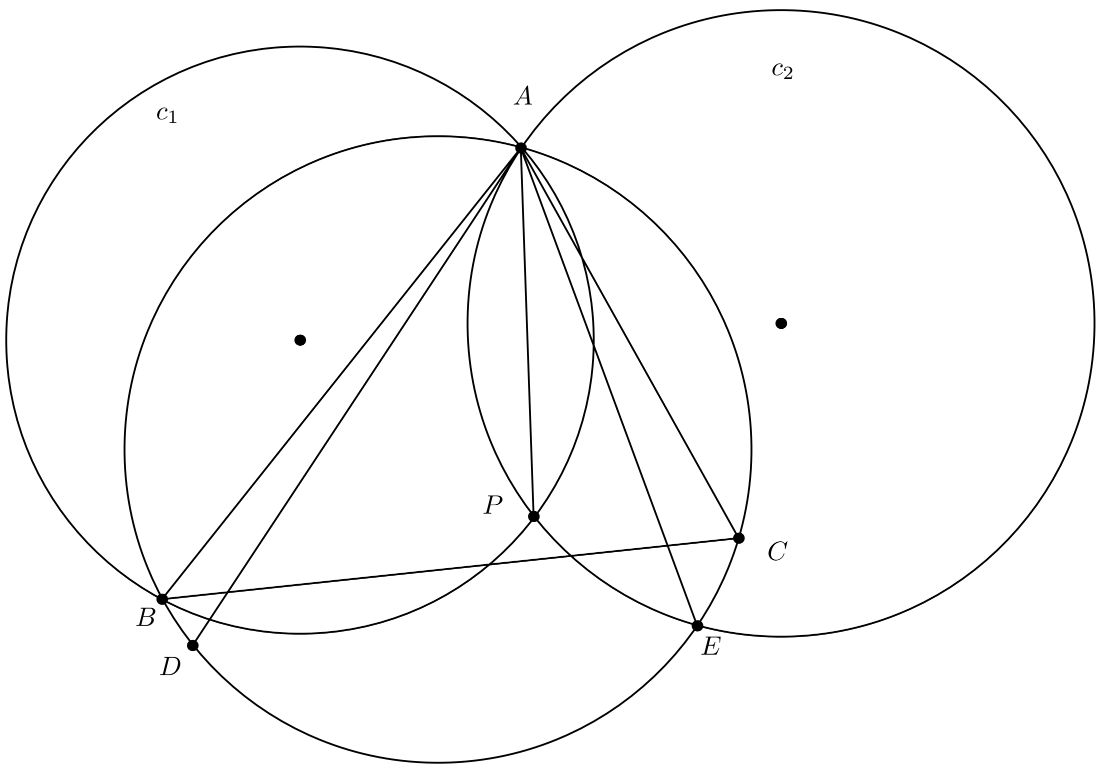
www.artofproblemsolving.com/community/c5235

by yunxiu, littletush

Day 1

---

1 In the triangle ABC, $\angle A$ is biggest. On the circumcircle of $\triangle ABC$, let D be the midpoint of $\widehat{ABC}$ and E be the midpoint of $\widehat{ACB}$. The circle c_1 passes through A, B and is tangent to AC at A, the circle c_2 passes through A, E and is tangent AD at A. c_1 and c_2 intersect at A and P. Prove that AP bisects $\angle BAC$.



---

2 Let p be a prime. We arrange the numbers in $\{1, 2, \ldots, p^2\}$ as a $p \times p$ matrix $A = (a_{ij})$. Next we can select any row or column and add 1 to every number in it, or subtract 1 from every number in it. We call the arrangement *good* if we can change every number of the matrix to 0 in a finite number of such moves. How many good arrangements are there?

---

3 Prove for any $M > 2$, there exists an increasing sequence of positive integers $a_1 < a_2 < \ldots$

satisfying:
1) $a_i > M^i$ for any i;
2) There exists a positive integer m and $b_1, b_2, \ldots, b_m \in \{-1, 1\}$, satisfying $n = a_1b_1 + a_2b_2 + \ldots + a_mb_m$ if and only if $n \in \mathbb{Z}/\{0\}$.

Day 2

1 Let $f(x) = (x+a)(x+b)$ where $a, b > 0$. For any reals $x_1, x_2, \ldots, x_n \geqslant 0$ satisfying $x_1 + x_2 + \ldots + x_n = 1$, find the maximum of $F = \sum_{1 \leqslant i < j \leqslant n} \min\{f(x_i), f(x_j)\}$.

2 Consider a square-free even integer n and a prime p, such that
1) $(n, p) = 1$;
2) $p \leq 2\sqrt{n}$;
3) There exists an integer k such that $p | n + k^2$.
Prove that there exists pairwise distinct positive integers a, b, c such that $n = ab + bc + ca$.

Proposed by Hongbing Yu

3 Find the smallest positive integer k such that, for any subset A of $S = \{1, 2, \ldots, 2012\}$ with $|A| = k$, there exist three elements x, y, z in A such that $x = a + b$, $y = b + c$, $z = c + a$, where a, b, c are in S and are distinct integers.

Proposed by Huawei Zhu