China National Olympiad 2013

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Day 1

1 Two circles K_1 and K_2 of different radii intersect at two points A and B, let C and D be two points on K_1 and K_2, respectively, such that A is the midpoint of the segment CD. The extension of DB meets K_1 at another point E, the extension of CB meets K_2 at another point F. Let l_1 and l_2 be the perpendicular bisectors of CD and EF, respectively.
i) Show that l_1 and l_2 have a unique common point (denoted by P).
ii) Prove that the lengths of CA, AP and PE are the side lengths of a right triangle.

2 Find all nonempty sets S of integers such that $3m-2n\in S$ for all (not necessarily distinct) $m,n\in S$.

3 Find all positive real numbers t with the following property: there exists an infinite set X of real numbers such that the inequality

$$\max\{|x-(a-d)|,|y-a|,|z-(a+d)|\}>td$$

holds for all (not necessarily distinct) $x,y,z\in X$, all real numbers a and all positive real numbers d.

Day 2

1 Let $n\geqslant 2$ be an integer. There are n finite sets $A_1,A_2,\ldots,A_n$ which satisfy the condition

$$|A_i\Delta A_j|=|i-j|\quad \forall i,j\in\{1,2,...,n\}.$$

Find the minimum of $\sum\limits_{i=1}^n |A_i|$.

2 For any positive integer n and $0\leqslant i\leqslant n$, denote $C_n^i\equiv c(n,i)\pmod 2$, where $c(n,i)\in\{0,1\}$. Define

$$f(n,q)=\sum_{i=0}^n c(n,i)q^i$$

where m,n,q are positive integers and $q+1\neq 2^\alpha$ for any $\alpha\in\mathbb{N}$. Prove that if $f(m,q)\,|\,f(n,q)$, then $f(m,r)\,|\,f(n,r)$ for any positive integer r.

3 Let m, n be positive integers. Find the minimum positive integer N which satisfies the following condition. If there exists a set S of integers that contains a complete residue system module m such that $|S| = N$, then there exists a nonempty set $A \subseteq S$ so that $n \mid \sum\limits_{x\in A} x$.