China National Olympiad 2014

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Day 1

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1 Let ABC be a triangle with $AB > AC$. Let D be the foot of the internal angle bisector of A. Points F and E are on AC, AB respectively such that B, C, F, E are concyclic. Prove that the circumcentre of DEF is the incentre of ABC if and only if $BE + CF = BC$.

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2 For the integer $n > 1$, define $D(n) = \{a - b \mid ab = n, a > b > 0, a, b \in \mathbb{N}\}$. Prove that for any integer $k > 1$, there exists pairwise distinct positive integers $n_1, n_2, \ldots, n_k$ such that $n_1, \ldots, n_k > 1$ and $|D(n_1) \cap D(n_2) \cap \cdots \cap D(n_k)| \geq 2$.

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3 Prove that: there exists only one function $f : \mathbb{N}^* \to \mathbb{N}^*$ satisfying:
i) $f(1) = f(2) = 1$;
ii)$f(n) = f(f(n-1)) + f(n - f(n-1))$ for $n \geq 3$.
For each integer $m \geq 2$, find the value of $f(2^m)$.

Day 2

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1 Let $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ be the prime factorisation of n. Define $\omega(n) = t$ and $\Omega(n) = a_1 + a_2 + \ldots + a_t$.
Prove or disprove:
For any fixed positive integer k and positive reals α, β, there exists a positive integer $n > 1$ such that
i) $\frac{\omega(n+k)}{\omega(n)} > \alpha$
ii) $\frac{\Omega(n+k)}{\Omega(n)} < \beta$.

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2 Let $f : X \to X$, where $X = \{1, 2, \ldots, 100\}$, be a function satisfying:
1) $f(x) \neq x$ for all $x = 1, 2, \ldots, 100$;
2) for any subset A of X such that $|A| = 40$, we have $A \cap f(A) \neq \emptyset$.
Find the minimum k such that for any such function f, there exist a subset B of X, where $|B| = k$, such that $B \cup f(B) = X$.

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3 For non-empty number sets S, T, define the sets $S + T = \{s + t \mid s \in S, t \in T\}$ and $2S = \{2s \mid s \in S\}$.
Let n be a positive integer, and A, B be two non-empty subsets of $\{1, 2 \ldots, n\}$. Show that there exists a subset D of $A + B$ such that
1) $D + D \subseteq 2(A + B)$,
2) $|D| \geq \frac{|A| \cdot |B|}{2n}$,

where $|X|$ is the number of elements of the finite set X.