China National Olympiad 2015

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Day 1

1 Let $z_1, z_2, ..., z_n$ be complex numbers satisfying $|z_i - 1| \le r$ for some r in $(0, 1)$. Show that

$$\left|\sum_{i=1}^{n} z_i\right| \cdot \left|\sum_{i=1}^{n} \frac{1}{z_i}\right| \ge n^2(1-r^2).$$

2 Let A, B, D, E, F, C be six points lie on a circle (in order) satisfy $AB = AC$.
Let $P = AD \cap BE, R = AF \cap CE, Q = BF \cap CD, S = AD \cap BF, T = AF \cap CD$.
Let K be a point lie on ST satisfy $\angle QKS = \angle ECA$.

Prove that $\frac{SK}{KT} = \frac{PQ}{QR}$

3 Let $n \ge 5$ be a positive integer and let A and B be sets of integers satisfying the following conditions:

i) $|A| = n$, $|B| = m$ and A is a subset of B
ii) For any distinct $x, y \in B$, $x + y \in B$ iff $x, y \in A$

Determine the minimum value of m.

Day 2

1 Determine all integers k such that there exists infinitely many positive integers n **not** satisfying

$$n + k | \binom{2n}{n}$$

2 Given 30 students such that each student has at most 5 friends and for every 5 students there is a pair of students that are not friends, determine the maximum k such that for all such possible configurations, there exists k students who are all not friends.

3 Let $a_1, a_2, ...$ be a sequence of non-negative integers such that for any m, n

$$\sum_{i=1}^{2m} a_{in} \le m.$$

Show that there exist k, d such that

$$\sum_{i=1}^{2k} a_{id} = k - 2014.$$