South East Mathematical Olympiad 2005

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Day 1 July 10th

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1 Let $a \in \mathbb{R}$ be a parameter.

(1) Prove that the curves of $y = x^2 + (a+2)x - 2a + 1$ pass through a fixed point; also, the vertices of these parabolas all lie on the curve of a certain parabola.

(2) If the function $x^2 + (a+2)x - 2a + 1 = 0$ has two distinct real roots, find the value range of the larger root.

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2 Circle C (with center O) does not have common point with line l. Draw OP perpendicular to l, $P \in l$. Let Q be a point on l (Q is different from P), QA and QB are tangent to circle C, and intersect the circle at A and B respectively. AB intersects OP at K. PM, PN are perpendicular to QB, QA, respectively, $M \in QB$, $N \in QA$. Prove that segment KP is bisected by line MN.

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3 Let n be positive integer, set $M = \{1, 2, \ldots, 2n\}$. Find the minimum positive integer k such that for any subset A (with k elements) of set M, there exist four pairwise distinct elements in A whose sum is $4n + 1$.

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4 Find all positive integer solutions (a, b, c) to the function $a^2 + b^2 + c^2 = 2005$, where $a \leq b \leq c$.

Day 2 July 11th

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5 Line l tangents unit circle S in point P. Point A and circle S are on the same side of l, and the distance from A to l is h ($h > 2$). Two tangents of circle S are drawn from A, and intersect line l at points B and C respectively.

Find the value of $PB \cdot PC$.

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6 Let $P(A)$ be the arithmetic-means of all elements of set $A = \{a_1, a_2, \ldots, a_n\}$, namely $P(A) = \frac{1}{n}\sum_{i=1}^{n} a_i$. We denote B "balanced subset" of A, if B is a non-empty subset of A and $P(B) = P(A)$. Let set $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Find the number of all "balanced subset" of M.

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7 (1) Find the possible number of roots for the equation $|x+1| + |x+2| + |x+3| = a$, where $x \in R$ and a is parameter.

(2) Let $\{a_1, a_2, \ldots, a_n\}$ be an arithmetic progression, $n \in \mathbb{N}$, and satisfy the condition

$$\sum_{i=1}^{n} |a_i| = \sum_{i=1}^{n} |a_i + 1| = \sum_{i=1}^{n} |a_i - 2| = 507.$$

Find the maximum value of n.

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8 Let $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$ and $\sin^3 \alpha + \sin^3 \beta + \sin^3 \gamma = 1$. Prove that

$$\tan^2 \alpha + \tan^2 \beta + \tan^2 \gamma \geq \frac{3\sqrt{3}}{2}.$$
