South East Mathematical Olympiad 2006

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Day 1

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1 Suppose $a > b > 0$, $f(x) = \dfrac{2(a+b)x+2ab}{4x+a+b}$. Show that there exists an unique positive number x, such that $f(x) = \left(\dfrac{a^{\frac{1}{3}}+b^{\frac{1}{3}}}{2}\right)^3$.

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2 In $\triangle ABC$, $\angle ABC = 90^\circ$. Points D, G lie on side AC. Points E, F lie on segment BD, such that $AE \perp BD$ and $GF \perp BD$. Show that if $BE = EF$, then $\angle ABG = \angle DFC$.

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3 There is a standard deck of 52 cards without jokers. The deck consists of four suits(diamond, club, heart, spade) which include thirteen cards in each. For each suit, all thirteen cards are ranked from 2 to A (i.e. $2, 3, \ldots, Q, K, A$). A pair of cards is called a *straight flush* if these two cards belong to the same suit and their ranks are adjacent. Additionally, "A" and "2" are considered to be adjacent (i.e. "A" is also considered as "1"). For example, spade A and spade 2 form a *straight flush*; diamond 10 and diamond Q are not a *straight flush* pair. Determine how many ways of picking thirteen cards out of the deck such that all ranks are included but no *straight flush* exists in them.

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4 Given any positive integer n, let a_n be the real root of equation $x^3 + \dfrac{x}{n} = 1$. Prove that
(1) $a_{n+1} > a_n$;
(2) $\sum_{i=1}^{n} \frac{1}{(i+1)^2 a_i} < a_n$.

Day 2

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1 [size=130]In $\triangle ABC, \angle A = 60^\circ$. $\odot I$ is the incircle of $\triangle ABC$. $\odot I$ is tangent to sides AB, AC at D, E, respectively. Line DE intersects line BI and CI at F, G respectively. Prove that [/size]$FG = \frac{BC}{2}$.

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2 Find the minimum value of real number m, such that inequality

$$m(a^3+b^3+c^3) \geq 6(a^2+b^2+c^2)+1$$

holds for all positive real numbers a, b, c where $a+b+c=1$.

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3 **(1)** Find the number of positive integer solutions (m, n, r) of the indeterminate equation $mn + nr + mr = 2(m+n+r)$.

(2) Given an integer $k(k>1)$, prove that indeterminate equation $mn+nr+mr=k(m+n+r)$ has at least $3k+1$ positive integer solutions (m,n,r).

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4 Given a circle with its perimeter equal to n($n\in N^*$), the least positive integer P_n which satisfies the following condition is called the *number of the partitioned circle*: there are P_n points ($A_1,A_2,\ldots,A_{P_n}$) on the circle; For any integer m ($1\le m\le n-1$), there always exist two points A_i,A_j ($1\le i,j\le P_n$), such that the length of arc A_iA_j is equal to m. Furthermore, all arcs between every two adjacent points A_i,A_{i+1} ($1\le i\le P_n$, $A_{p_n+1}=A_1$) form a sequence $T_n=(a_1,a_2,,,a_{p_n})$ called the *sequence of the partitioned circle*. For example when $n=13$, the number of the partitioned circle P_{13}=4, the sequence of the partitioned circle $T_{13}=(1,3,2,7)$ or $(1,2,6,4)$. Determine the values of P_{21} and P_{31}, and find a possible solution of T_{21} and T_{31} respectively.
