South East Mathematical Olympiad 2007
www.artofproblemsolving.com/community/c5246
by jred

Day 1

1 Determine the number of real number a, such that for every a, equation $x^3 = ax + a + 1$ has a root x_0 satisfying following conditions:
(a) x_0 is an even integer;
(b) $|x_0| < 1000$.

2 AB is the diameter of semicircle O. C,D are two arbitrary points on semicircle O. Point P lies on line CD such that line PB is tangent to semicircle O at B. Line PO intersects line CA, AD at point E, F respectively. Prove that OE=OF.

3 Let $a_i = min\{k + \frac{i}{k} | k \in N^*\}$, determine the value of $S_{n^2} = [a_1] + [a_2] + \cdots + [a_{n^2}]$, where $n \geq 2$. ($[x]$ denotes the greatest integer not exceeding x)

4 A sequence of positive integers with n terms satisfies $\sum_{i=1}^{n} a_i = 2007$. Find the least positive integer n such that there exist some consecutive terms in the sequence with their sum equal to 30.

Day 2

1 Let $f(x)$ be a function satisfying $f(x+1) - f(x) = 2x + 1 (x \in \mathbb{R})$.In addition, $|f(x)| \leq 1$ holds for $x \in [0, 1]$. Prove that $|f(x)| \leq 2 + x^2$ holds for $x \in \mathbb{R}$.

2 In right-angle triangle ABC, $\angle C = 90$, Point D is the midpoint of side AB. Points M and C lie on the same side of AB such that $MB \perp AB$, line MD intersects side AC at N, line MC intersects side AB at E. Show that $\angle DBN = \angle BCE$.

3 Find all triples (a, b, c) satisfying the following conditions:
(i) a, b, c are prime numbers, where $a < b < c < 100$.
(ii) $a + 1, b + 1, c + 1$ form a geometric sequence.

4 Let a,b,c be positive real numbers satisfying $abc = 1$. Prove that inequality $\frac{a^k}{a+b} + \frac{b^k}{b+c} + \frac{c^k}{c+a} \geq \frac{3}{2}$ holds for all integer k ($k \geq 2$).