South East Mathematical Olympiad 2008

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by jred

Day 1

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1 Given a set $S=\{1,2,3,\ldots,3n\},(n\in N^*)$, let T be a subset of S, such that for any $x,y,z\in T$ (not necessarily distinct) we have $x+y+z\notin T$. Find the maximum number of elements T can have.

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2 Let $\{a_n\}$ be a sequence satisfying: $a_1=1$ and $a_{n+1}=2a_n+n\cdot(1+2^n),(n=1,2,3,\cdots)$. Determine the general term formula of $\{a_n\}$.

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3 In $\triangle ABC$, side $BC>AB$. Point D lies on side AC such that $\angle ABD=\angle CBD$. Points Q,P lie on line BD such that $AQ\bot BD$ and $CP\bot BD$. M,E are the midpoints of side AC and BC respectively. Circle O is the circumcircle of $\triangle PQM$ intersecting side AC at H. Prove that O,H,E,M lie on a circle.

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4 Let m,n be positive integers $(m,n>=2)$. Given an n-element set A of integers $(A=\{a_1,a_2,\cdots,a_n\})$, for each pair of elements $a_i,a_j(j>i)$, we make a difference by a_j-a_i. All these C_n^2 differences form an ascending sequence called derived sequence of set A. Let $\bar{A}$ denote the derived sequence of set A. Let $\bar{A}(m)$ denote the number of terms divisible by m in $\bar{A}$. Prove that $\bar{A}(m)\geq\bar{B}(m)$ where $A=\{a_1,a_2,\cdots,a_n\}$ and $B=\{1,2,\cdots,n\}$.

Day 2

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1 Let λ be a positive real number. Inequality $|\lambda xy+yz|\leq\dfrac{\sqrt{5}}{2}$ holds for arbitrary real numbers x,y,z satisfying $x^2+y^2+z^2=1$. Find the maximal value of λ.

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2 Circle I is the incircle of $\triangle ABC$. Circle I is tangent to sides BC and AC at M,N respectively. E,F are midpoints of sides AB and AC respectively. Lines EF,BI intersect at D. Show that M,N,D are collinear.

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3 Captain Jack and his pirate men plundered six chests of treasure $(A_1,A_2,A_3,A_4,A_5,A_6)$. Every chest A_i contains a_i coins of gold, and all a_is are pairwise different $(i=1,2,\cdots,6)$. They place all chests according to a layout (see the attachment) and start to alternately take out one chest a time between the captain and a pirate who serves as the delegate of the captains men. A rule must be complied with during the game: only those chests that are not adjacent to other two or more chests are allowed to be taken out. The captain will win the game if the coins of gold he obtains are not less than those of his men in the end. Let the captain be granted to

take chest firstly, is there a certain strategy for him to secure his victory?

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4 Let n be a positive integer. $f(n)$ denotes the number of n-digit numbers $\overline{a_1a_2\cdots a_n}$(wave numbers) satisfying the following conditions :
(i) for each $a_i \in \{1,2,3,4\}$, $a_i \neq a_{i+1}$, $i=1,2,\cdots$;
(ii) for $n \geq 3$, $(a_i - a_{i+1})(a_{i+1} - a_{i+2})$ is negative, $i=1,2,\cdots$.
(1) Find the value of $f(10)$;
(2) Determine the remainder of $f(2008)$ upon division by 13.
