South East Mathematical Olympiad 2010

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Day 1

1	Let $a,b,c \in \{0,1,2,\cdots,9\}$.The quadratic equation $ax^2+bx+c=0$ has a rational root. Prove that the three-digit number abc is not a prime number.
2	For any set $A=\{a_1,a_2,\cdots,a_m\}$, let $P(A)=a_1a_2\cdots a_m$. Let $n=\binom{2010}{99}$, and let $A_1, A_2,\cdots,A_n$ be all 99-element subsets of $\{1,2,\cdots,2010\}$. Prove that $2010 \| \sum_{i=1}^{n} P(A_i)$.
3	The incircle of triangle ABC touches BC at D and AB at F, intersects the line AD again at H and the line CF again at K. Prove that $\frac{FD\times HK}{FH\times DK}=3$
4	Let a and b be positive integers such that $1\le a<b\le 100$. If there exists a positive integer k such that $ab\|a^k+b^k$, we say that the pair (a,b) is good. Determine the number of good pairs.

Day 2

1	ABC is a triangle with a right angle at C. M_1 and M_2 are two arbitrary points inside ABC, and M is the midpoint of M_1M_2. The extensions of BM_1, BM and BM_2 intersect AC at N_1, N and N_2 respectively. Prove that $\frac{M_1N_1}{BM_1}+\frac{M_2N_2}{BM_2}\geq 2\frac{MN}{BM}$
2	Let $\mathbb{N}^*$ be the set of positive integers. Define $a_1=2$, and for $n=1,2,\ldots,$ $$a_{n+1}=\min\{\lambda\|\frac{1}{a_1}+\frac{1}{a_2}+\cdots\frac{1}{a_n}+\frac{1}{\lambda}<1, \lambda\in\mathbb{N}^*\}$$ Prove that $a_{n+1}=a_n^2-a_n+1$ for $n=1,2,\ldots$.
3	Let n be a positive integer. The real numbers $a_1,a_2,\cdots,a_n$ and $r_1,r_2,\cdots,r_n$ are such that $a_1\le a_2\le\cdots\le a_n$ and $0\le r_1\le r_2\le\cdots\le r_n$. Prove that $\sum_{i=1}^{n}\sum_{j=1}^{n}a_ia_j\min(r_i,r_j)\ge 0$
4	$A_1, A_2,\cdots,A_8$ are fixed points on a circle. Determine the smallest positive integer n such that among any n triangles with these eight points as vertices, two of them will have a common side.