South East Mathematical Olympiad 2013

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Day 1 July 27th

1 Let a, b be real numbers such that the equation $x^3 - ax^2 + bx - a = 0$ has three positive real roots . Find the minimum of $\frac{2a^3-3ab+3a}{b+1}$.

2 $\triangle ABC$, $AB > AC$. the incircle I of $\triangle ABC$ meet BC at point D, AD meet I again at E. EP is a tangent of I, and EP meet the extension line of BC at P. $CF \parallel PE$, $CF \cap AD = F$. the line BF meet I at M, N, point M is on the line segment BF, the line segment PM meet I again at Q. Show that $\angle ENP = \angle ENQ$

3 A sequence $\{a_n\}$, $a_1 = 1, a_2 = 2, a_{n+1} = \dfrac{a_n^2 + (-1)^n}{a_{n-1}}$. Show that $a_m^2 + a_{m+1}^2 \in \{a_n\}, \forall m \in \mathbb{N}$

4 There are 12 acrobats who are assigned a distinct number ($1, 2, \cdots, 12$) respectively. Half of them stand around forming a circle (called circle A); the rest form another circle (called circle B) by standing on the shoulders of every two adjacent acrobats in circle A respectively. Then circle A and circle B make up a formation. We call a formation a *tower* if the number of any acrobat in circle B is equal to the sum of the numbers of the two acrobats whom he stands on. How many heterogeneous *towers* are there?
(Note: two *towers* are homogeneous if either they are symmetrical or one may become the other one by rotation. We present an example of 8 acrobats (see attachment). Numbers inside the circle represent the circle A; numbers outside the circle represent the circle B. All these three formations are *towers*, however they are homogeneous *towers*.)

Day 2 July 28th

5 $f(x) = \sum\limits_{i=1}^{2013} \left[\dfrac{x}{i!}\right]$. A integer n is called *good* if $f(x) = n$ has real root. How many good numbers are in $\{1, 3, 5, \dots, 2013\}$?

6 $n > 1$ is an integer. The first n primes are $p_1 = 2, p_2 = 3, \dots, p_n$. Set $A = p_1^{p_1} p_2^{p_2} ... p_n^{p_n}$.
Find all positive integers x, such that $\dfrac{A}{x}$ is even, and $\dfrac{A}{x}$ has exactly x divisors

7 Given a 3×3 grid, we call the remainder of the grid an *angle* when a 2×2 grid is cut out from the grid. Now we place some *angles* on a 10×10 grid such that the borders of those *angles* must lie on the grid lines or its borders, moreover there is no overlap among the *angles*. Determine the maximal value of k, such that no matter how we place k *angles* on the grid, we can always place another *angle* on the grid.

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8 $n \geq 3$ is a integer. $\alpha, \beta, \gamma \in (0,1)$. For every $a_k, b_k, c_k \geq 0 (k = 1, 2, \ldots, n)$ with $\sum\limits_{k=1}^{n}(k+\alpha)a_k \leq \alpha$, $\sum\limits_{k=1}^{n}(k+\beta)b_k \leq \beta$, $\sum\limits_{k=1}^{n}(k+\gamma)c_k \leq \gamma$, we always have $\sum\limits_{k=1}^{n}(k+\lambda)a_k b_k c_k \leq \lambda$.
Find the minimum of λ
