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1 Let $X,\ Y$ and Z be the midpoints of sides $BC,\ CA$, and AB of the triangle ABC, respectively. Let P be a point inside the triangle. Prove that the quadrilaterals $AZPY,\ BXPZ$, and $CYPX$ have equal areas if, and only if, P is the centroid of ABC.

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2 Prove that if a, b, c are positive real numbers, then the least possible value of

$$6a^3 + 9b^3 + 32c^3 + \frac{1}{4abc}$$

is 6. For which values of a, b and c is equality attained?

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3 Consider a 3×7 grid of squares. Each square may be coloured green or white.

(a) Is it possible to find a colouring so that no subrectangle has all four corner squares of the same colour?
(b) Is it possible for a 4×6 grid?

Subrectangles must have their corners at grid-points of the original diagram. The corner squares of a subrectangle must be different. The original diagram is a subrectangle of itself.

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4 Fawzi cuts a spherical cheese completely into (at least three) slices of equal thickness. He starts at one end, making successive parallel cuts, working through the cheese until the slicing is complete. The discs exposed by the first two cuts have integral areas.

(i) Prove that all the discs that he cuts have integral areas.
(ii) Prove that the original sphere had integral surface area if, and only if, the area of the second disc that he exposes is even.
