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by Malik

1 Let $a_1, a_2, \ldots, a_{2n}$ be positive real numbers such that $a_j a_{n+j} = 1$ for the values $j = 1, 2, \ldots, n$.

a. Prove that either the average of the numbers $a_1, a_2, \ldots, a_n$ is at least 1 or the average of the numbers $a_{n+1}, a_{n+2}, \ldots, a_{2n}$ is at least 1.

b. Assuming that $n \geq 2$, prove that there exist two distinct numbers j, k in the set $\{1, 2, \ldots, 2n\}$ such that

$$|a_j - a_k| < \frac{1}{n-1}.$$

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2 In triangle ABC, the bisector of angle B meets the opposite side AC at B'. Similarly, the bisector
of angle C meets the opposite side AB at C'. Prove that $A = 60^\circ$ if, and only if, $BC' + CB' = BC$.

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3 There are n people standing on a circular track. We want to perform a number of *moves* so that we end up with a situation where the distance between every two neighbours is the same. The *move* that is allowed consists in selecting two people and asking one of them to walk a distance d on the circular track clockwise, and asking the other to walk the same distance on the track anticlockwise. The two people selected and the quantity d can vary from move to move.

Prove that it is possible to reach the desired situation (where the distance between every two neighbours is the same) after at most $n-1$ moves.

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4 Let m, n be integers. It is known that there are integers a, b such that $am + bn = 1$ if, and only if, the greatest common divisor of m, n is 1. *You are not required to prove this*.

Now suppose that p, q are different odd primes. In each case determine if there are integers a, b such that $ap + bq = 1$ so that the given condition is satisfied:

a. p divides b and q divides a;
b. p divides a and q divides b;
c. p does not divide a and q does not divide b.
